ON THE LOG-CONCAVITY OF THE DEGENERATE BERNOULLI NUMBERS

FLORIAN LUCA
Instituto de Matemáticas
Universidad Nacional Autónoma de México
C.P. 58089, Morelia, Michoacán, México
and
The John Knopfmacher Centre
for Applicable Analysis and Number Theory
University of the Witwatersrand
P.O. Wits 2050, South Africa
`fluca@matmor.unam.mx`

PAUL THOMAS YOUNG
Department of Mathematics
College of Charleston
Charleston, SC 29424, USA
`paul@math.cofc.edu`

September 22, 2011

Abstract

The degenerate Bernoulli numbers $\beta_n(\lambda)$ are polynomials with rational coefficients of degree n in the variable λ, which arise in several combinatorial settings. An appropriate change of variable transforms $\beta_n(\lambda)$ into a polynomial whose coefficients are all positive. Here, we prove that this transformed polynomial is log-concave, and therefore unimodal. As a consequence, we deduce bounds on the absolute values of the roots of $\beta_n(\lambda)$.

1 Introduction

Carlitz [3] defined the *degenerate Bernoulli numbers* $\beta_n(\lambda)$ for $\lambda \neq 0$ by means of the generating function

$$\frac{t}{(1+\lambda t)^{\mu}-1} = \sum_{n=0}^{\infty} \beta_n(\lambda)\frac{t^n}{n!},$$

where $\lambda\mu = 1$. Each $\beta_n(\lambda)$ is a polynomial of degree n in λ with rational coefficients; for $n > 1$ the polynomial $\beta_n(\lambda)$ is an even (resp. odd) function of λ when n is even (resp. odd). These polynomials occur in expressions for sums of falling factorials [3, 9], for divided differences of binomial coefficients [1], for coset products in factor rings [10], and in game theory probabilities [5]. Since $(1+\lambda t)^{\mu} \to e^t$ as $\lambda \to 0$, it is evident that $\beta_n(0) = B_n$, the n-th *Bernoulli number*, which is defined by

$$\frac{t}{e^t-1} = \sum_{n=0}^{\infty} B_n\frac{t^n}{n!};$$

since $((1+t)^{\mu}-1)/\mu \to \log(1+t)$ as $\mu \to 0$, we have $\lim_{\lambda\to\infty} \lambda^{-n}\beta_n(\lambda) = n!\, b_n$, where the *Bernoulli numbers of the second kind* b_n are defined by

$$\frac{t}{\log(1+t)} = \sum_{n=0}^{\infty} b_n t^n.$$

Thus, $\beta_n(\lambda)$ is a polynomial of degree n in λ whose constant term is B_n and whose leading coefficient is $n!\, b_n$.

For $n > 1$, Howard [6] gave an explicit formula for the coefficients of $\beta_n(\lambda)$,

$$\beta_n(\lambda) = n! b_n \lambda^n + \sum_{k=1}^{[n/2]} \frac{nB_{2k}}{2k} s(n-1, 2k-1)\lambda^{n-2k}, \tag{1}$$

involving the *Stirling numbers $s(n,k)$ of the first kind*, which may be defined by

$$x(x-1)\cdots(x-n+1) = \sum_{k=0}^{n} s(n,k)x^k. \tag{2}$$

The first few values of $\beta_n(\lambda)$ are

$$\beta_0(\lambda) = 1, \qquad \beta_1(\lambda) = -\frac{1}{2} + \frac{1}{2}\lambda, \qquad \beta_2(\lambda) = \frac{1}{6} - \frac{1}{6}\lambda^2,$$

$$\beta_3(\lambda) = -\frac{1}{4}\lambda + \frac{1}{4}\lambda^3, \qquad \beta_4(\lambda) = -\frac{1}{30} + \frac{2}{3}\lambda^2 - \frac{19}{30}\lambda^4,$$
$$\beta_5(\lambda) = \frac{1}{4}\lambda - \frac{5}{2}\lambda^3 + \frac{9}{4}\lambda^5, \qquad \beta_6(\lambda) = \frac{1}{42} - \frac{7}{4}\lambda^2 + 12\lambda^4 - \frac{863}{84}\lambda^6,$$
$$\beta_7(\lambda) = -\frac{5}{12}\lambda + \frac{105}{8}\lambda^3 - 70\lambda^5 + \frac{1375}{24}\lambda^7,$$
$$\beta_8(\lambda) = -\frac{1}{30} + \frac{50}{9}\lambda^2 - \frac{1624}{15}\lambda^4 + 480\lambda^6 - \frac{33953}{90}\lambda^8,$$
$$\beta_9(\lambda) = \frac{21}{20}\lambda - 70\lambda^3 + \frac{9849}{10}\lambda^5 - 3780\lambda^7 + \frac{57281}{20}\lambda^9.$$

It is easy to verify that for even n the polynomial $\beta_n(\lambda)$ given by (1) is a polynomial in λ^2 whose nonzero coefficients alternate in sign, as is $\beta_n(\lambda)/\lambda$ if $n > 1$ is odd. Here, we consider the transformed polynomial $\alpha_n(x)$ defined for $n > 1$ by

$$\alpha_n(x) := n!|b_n| + \sum_{k=1}^{[n/2]} \frac{n|B_{2k}|}{2k} |s(n-1, 2k-1)| x^k, \tag{3}$$

which is a polynomial of degree $[n/2]$ in x having all coefficients positive, in agreement up to sign, in reverse order, with the nonzero coefficients of $\beta_n(\lambda)$. The polynomials shown at (1) and (3) are related by the change of variables

$$\beta_n(\lambda) = -(-\lambda)^n \alpha_n(-1/\lambda^2), \quad \alpha_n(x) = -\left(-\sqrt{-x}\right)^n \beta_n(\sqrt{-1/x}). \tag{4}$$

A polynomial $p(x) = a_0 + a_1 x + \cdots + a_m x^m$ with positive real coefficients a_i is called *log-concave* if its sequence of coefficients $\{a_0, a_1, ..., a_m\}$ is logarithmically concave, meaning that $a_{i-1}a_{i+1} \leqslant a_i^2$ for $0 < i < m$. The main result of this paper is the following:

Theorem 1. *For all $n > 1$, the polynomial $\alpha_n(x)$ is log-concave.*

It is well-known that any log-concave polynomial is *unimodal*, meaning that there exists an index j for which $a_0 \leqslant a_1 \leqslant \cdots \leqslant a_j \geqslant a_{j+1} \geqslant \cdots \geqslant a_m$. We use the Theorem 1 to deduce the following bounds for the roots of $\beta_n(\lambda)$.

Corollary 2. *For all even $n > 1$, if $\lambda \in \mathbb{C}$ is a root of the polynomial $\beta_n(\lambda)$, then*

$$\frac{\sqrt{2}}{\pi(n-1)} \leqslant |\lambda| \leqslant 1 + \log(n-1),$$

and for all odd $n > 1$, if $\lambda \in \mathbb{C}$ is a nonzero root of the polynomial $\beta_n(\lambda)$, then

$$\frac{\sqrt{6}}{\pi(n-2)} \leqslant |\lambda| \leqslant 1 + \log(n-1).$$

It is well known that $\lambda = \pm 1$ is a root of $\beta_n(\lambda)$ for all $n > 1$, and it was recently shown in [9], that if n is odd, then $\lambda = \pm 1/d$ is a root of $\beta_n(\lambda)$ for every divisor d of $n-2$; so, in particular, for odd n we have rational roots at $\lambda = \pm 1/(n-2)$ and $\lambda = \pm 1$. A primary motivation for this paper was to examine the size of the other roots of $\beta_n(\lambda)$. Based on numerical computations, the lower bounds given in Corollary 2 appear to be fairly sharp. On the other hand, for $n \leqslant 100$ the roots of $\beta_n(\lambda)$ all satisfy $|\lambda| \leqslant 1$, which is quite a bit smaller than our stated upper bound. We conjecture that all roots λ of all the polynomials $\beta_n(\lambda)$ have absolute values $|\lambda| \leqslant 1$.

2 Estimates for the Bernoulli numbers of the second kind

In order to prove our results, we need some estimates on the growth of the sequence $\{b_n\}_{n \geqslant 1}$. It is well-known [9, 10], that the Bernoulli numbers of the second kind b_n may also be defined by

$$n! b_n = \int_0^1 x(x-1)\cdots(x-n+1)dx. \tag{5}$$

Lemma 3. *We have*

$$\frac{1}{8n(1+\log(n-1))^2} < |b_n| < \frac{1}{n(\log n)^2} \tag{6}$$

for all $n \geqslant 2$.

Proof. The inequalities may be verified directly for $n = 2, 3, 4$. Considering tangent lines and secant lines for $f(t) = e^{-t}$, we have the inequalities

$$1 - t \leqslant e^{-t} \leqslant 1 - rt \quad \text{for} \quad t \in [0,1], \tag{7}$$

where $r := 1 - e^{-1}$. Considering the upper and lower approximations by rectangles to the integral of $g(t) = 1/t$ on the interval $[1, n]$, we have

$$\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} < \log n < 1 + \frac{1}{2} + \cdots + \frac{1}{n-1}. \tag{8}$$

By the integral formula (5) for b_n and the left inequality in (7), we have

$$\begin{aligned} n!|b_n| &= \int_0^1 x(1-x)(2-x)\cdots(n-1-x)dx \\ &= (n-1)!\int_0^1 x\left(1-\frac{x}{1}\right)\left(1-\frac{x}{2}\right)\cdots\left(1-\frac{x}{n-1}\right)dx \\ &< (n-1)!\int_0^1 x\exp\left(-x\left(1+\frac{1}{2}+\cdots+\frac{1}{n-1}\right)\right)dx. \end{aligned} \tag{9}$$

Then, by the left inequality in (8), we have

$$-x\left(1+\frac{1}{2}+\cdots+\frac{1}{n-1}\right) < -x\log n \quad \text{for all} \quad x\in(0,1]. \tag{10}$$

Inserting estimate (10) into (9), we get

$$n!|b_n| < (n-1)!\int_0^1 x\exp(-x\log n)dx.$$

With the change of variable $u := x\log n$ in the above integral, for which $du = (\log n)dx$, we have that

$$\begin{aligned} n!|b_n| &< \frac{(n-1)!}{(\log n)^2}\int_0^{\log n} ue^{-u}du = \frac{(n-1)!}{(\log n)^2}\left(-ue^{-u}-e^{-u}\Big|_{u=0}^{u=\log n}\right) \\ &= \frac{(n-1)!}{(\log n)^2}\left(1-\frac{1+\log n}{n}\right) < \frac{(n-1)!}{(\log n)^2}, \end{aligned}$$

which implies the right inequality in (6).

Again by the integral formula (5) and the right inequality in (7), we deduce, with $x =: rt$, that

$$\begin{aligned} n!|b_n| &= \int_0^1 x(1-x)(2-x)\cdots(n-1-x)dx \\ &= (n-1)!\int_0^1 x\left(1-\frac{x}{1}\right)\left(1-\frac{x}{2}\right)\cdots\left(1-\frac{x}{n-1}\right)dx \\ &= (n-1)!\int_0^{r^{-1}} r^2t\left(1-\frac{rt}{1}\right)\left(1-\frac{rt}{2}\right)\cdots\left(1-\frac{rt}{n-1}\right)dt \\ &> (n-1)!\int_0^{r^{-1}} r^2t\exp\left(-t\left(1+\frac{1}{2}+\cdots+\frac{1}{n-1}\right)\right)dt. \end{aligned} \tag{11}$$

Then, by the right inequality in (8), we have

$$-x\left(1+\frac{1}{2}+\cdots+\frac{1}{n-1}\right) > -x(1+\log(n-1)) \quad \text{for all} \quad x \in (0,1]. \tag{12}$$

Inserting estimate (12) into (11), we get

$$n!|b_n| > (n-1)! \int_0^{r^{-1}} r^2 t \exp(-t(1+\log(n-1)))dt.$$

Integration by parts above then yields

$$\begin{aligned} n|b_n| &> \left(\frac{-re^{-(1+\log(n-1))}}{1+\log(n-1)} - \frac{r^2 e^{-(1+\log(n-1))}}{(1+\log(n-1))^2} + \frac{r^2}{(1+\log(n-1))^2}\right) \\ &= \frac{r^2}{(1+L)^2}\left(1-\frac{e^{-1}}{n-1}\left(1+\frac{1+L}{r}\right)\right), \end{aligned}$$

where $L := \log(n-1)$.

Since it is easily established that $(1+L)/(n-1) \leqslant 1$ for all n (in fact, this follows from the left inequality (7) with $t := -\log(n-1)$), we have

$$n|b_n| > \frac{r^2}{(1+L)^2}\left(1-\frac{e^{-1}}{n-1}-\frac{e^{-1}}{1-e^{-1}}\right).$$

Therefore, when $n > 4$, we have

$$n|b_n| > \frac{1}{8(1+L)^2},$$

giving the left inequality of (6) and completing the proof. ☐

3 The log-concavity of $(s-1)\zeta(s)$

To prove our Theorem 1, we need to verify that $(s-1)\zeta(s)$ is log-concave for sufficiently large real numbers $s > 1$, where $\zeta(s)$ is the Riemann zeta function. We begin with the following elementary estimates.

Lemma 4. *We have* $s\zeta(s) \leqslant s+1$ *for* $s \geqslant 3$, *and* $s^2\zeta(s) \leqslant s^2+1$ *for* $s \geqslant 5$.

Proof. Using the series for $\zeta(s)$ as an approximation to an integral with rectangles from below, we get

$$\begin{aligned}\zeta(s) &= 1 + 2^{-s} + 3^{-s} + \cdots \\ &< 1 + 2^{-s} + \int_2^\infty x^{-s}\,dx \\ &= 1 + 2^{-s} + \frac{2^{1-s}}{s-1} \\ &\leqslant 1 + \frac{1}{s} \quad \text{for} \quad s \geqslant 3,\end{aligned}$$

giving the first statement. Since

$$1 + 2^{-s} + \frac{2^{1-s}}{s-1} \leqslant 1 + \frac{1}{s^2} \quad \text{for} \quad s \geqslant 5,$$

the second statement follows as well.

□

Let

$$f(s) := \log((s-1)\zeta(s)) \quad \text{for} \quad s > 1.$$

Lemma 5. *We have* $f''(s) < 0$ *for all* $s \geqslant 6$.

Proof. We first observe that

$$\begin{aligned}f'(s) &= \frac{d}{ds}\left(\log(s-1) + \log\zeta(s)\right) = \frac{1}{s-1} + \frac{\zeta'(s)}{\zeta(s)} \\ &= \frac{1}{s-1} - \sum_{n=2}^{\infty} \Lambda(n) n^{-s},\end{aligned}$$

where

$$\Lambda(n) = \begin{cases} \log p, & \text{if } n = p^m \text{ for some prime } p \text{ and integer } m \geqslant 1, \\ 0, & \text{otherwise} \end{cases}$$

is the von–Mangoldt function, and therefore

$$f''(s) = -\frac{1}{(s-1)^2} + \sum_{n=2}^{\infty} \Lambda(n)(\log n) n^{-s}$$

for all $s > 1$. Since $0 \leqslant \Lambda(n) \leqslant \log n < n^{1/2}$ for $n \geqslant 1$, we have by Lemma 4

$$\begin{aligned} f''(s) &= -\frac{1}{(s-1)^2} + \sum_{n=2}^{\infty} \Lambda(n)(\log n) n^{-s} \\ &< -\frac{1}{(s-1)^2} + \sum_{n=2}^{\infty} n \cdot n^{-s} \qquad (\text{for} \quad s > 2) \\ &= -\frac{1}{(s-1)^2} + (\zeta(s-1) - 1) \\ &\leqslant -\frac{1}{(s-1)^2} + \frac{1}{(s-1)^2} = 0 \quad \text{for} \quad s \geqslant 6, \end{aligned}$$

as desired. □

Remark. While not needed for our purposes, we can verify numerically that in fact $f''(s) < 0$ for all $s > 1$. To estimate $f''(s)$ for $s > 1$ and near 1, we use the formula ([2], p. 56)

$$\zeta(s) = \frac{s}{s-1} - s \int_1^{\infty} \frac{\{x\}}{x^{s+1}} dx := \frac{s}{s-1} - s g(s),$$

where $\{x\} = x - [x]$ denotes the fractional part of x and

$$g(s) := \int_1^{\infty} \frac{\{x\}}{x^{s+1}} dx.$$

Then

$$f(s) = \log((s-1)\zeta(s)) = \log(s - s(s-1)g(s)) = \log s + \log(1 - (s-1)g(s)).$$

For $1 < s < 2$, we have that

$$0 < (s-1)g(s) = (s-1) \int_1^{\infty} \frac{\{x\}}{x^{s+1}} dx < (s-1) \int_1^{\infty} \frac{dx}{x^2} = s - 1 < 1,$$

so we can expand the logarithm in series getting

$$f(s) = \log s - \sum_{k \geqslant 1} \frac{1}{k}(s-1)^k g(s)^k = \log s - (s-1)g(s) - \frac{1}{2}(s-1)^2 g(s)^2 + (s-1)^3 h(s),$$

where $h(s)$ is analytic in a neighborhood of 1. Taking derivatives we get

$$f'(s) = \frac{1}{s} - g(s) - (s-1)g'(s) - (s-1)g(s)^2 + (s-1)^2 a(s),$$

where $a(s) := -g(s)g'(s) + 3h(s) + (s-1)h'(s)$. So,

$$f''(s) = -\frac{1}{s^2} - 2g'(s) - g(s)^2 + (s-1)b(s),$$

where $b(s) := -g''(s) - 2g(s)g'(s) + 2a(s) + (s-1)a'(s)$. Thus, $f''(s)$ in a neighborhood of 1 is close to

$$-1 - 2g'(1) - g(1)^2 = -1 + 2\int_1^\infty \frac{\{x\}\log x}{x^2}dx - \left(\int_1^\infty \frac{\{x\}}{x^2}dx\right)^2,$$

a number which is about -0.19.

4 The log-concavity of $\alpha_n(x)$

In this section, we give the proof of Theorem 1. Log-concavity of $\alpha_n(x)$ is easily verified directly for $n \leqslant 4$, so in what follows we assume $n \geqslant 4$. The polynomial $\alpha_n(x) = a_0 + a_1 x + \cdots + a_m x^m$ has degree $m := [n/2]$, with coefficients $a_0 := n!|b_n|$ and $a_k := (n|B_{2k}|/2k)|s(n-1, 2k-1)|$ for $1 \leqslant k \leqslant m$. Log-concavity of $\alpha_n(x)$ is equivalent to the statement that the sequence of ratios of coefficients

$$\frac{a_0}{a_1}, \frac{a_1}{a_2}, \ldots, \frac{a_{m-1}}{a_m} \tag{13}$$

is nondecreasing.

For $\alpha_n(x)$ we have, by Lemma 3, that

$$\frac{a_0}{a_1} = \frac{n!|b_n|}{(n/2)|B_2 s(n-1,1)|} = 12(n-1)|b_n| < \frac{12}{(\log n)^2}. \tag{14}$$

By direct calculation, we have

$$\frac{a_1}{a_2} = \frac{4|B_2 s(n-1,1)|}{2|B_4 s(n-1,3)|} = 10\frac{|s(n-1,1)|}{|s(n-1,3)|}. \tag{15}$$

From equation (4.8) in [9], we have

$$\begin{aligned}
|s(n-1,3)| &= |s(n-1,1)| \sum_{1\leqslant i<j\leqslant n-2} \frac{1}{ij} \\
&= \frac{|s(n-1,1)|}{2}\left(\left(\sum_{k=1}^{n-2}\frac{1}{k}\right)^2 - \sum_{k=1}^{n-2}\frac{1}{k^2}\right) \\
&< \frac{|s(n-1,1)|}{2}((1+\log(n-2))^2-1), \qquad (16)
\end{aligned}$$

where the last inequality above follows by using the left inequality of (8). Therefore, by estimates (14), (15), and (16), we have

$$\frac{a_0}{a_1} < \frac{12}{(\log n)^2} < \frac{20}{(1+\log(n-2))^2-1} < \frac{a_1}{a_2},$$

thus demonstrating the first inequality required in (13).

The log-concavity of $\alpha_n(x)$ will then be demonstrated by showing that the sequence

$$\left\{\frac{a_k}{a_{k+1}}\right\} = \left\{\frac{(2k+2)|B_{2k}s(n-1,2k-1)|}{2k|B_{2k+2}s(n-1,2k+1)|}\right\}$$

for $1 \leqslant k < m$ is nondecreasing. From Theorem 3.1 in [8], we know that the sequence

$$\left\{\frac{1}{2k(2k-1)}\frac{|s(n-1,2k-1)|}{|s(n-1,2k+1)|}\right\}_{k\geqslant 1}$$

is an increasing sequence in k. It therefore suffices to show that the sequence

$$\left\{\frac{(2k+2)|B_{2k}|}{2k|B_{2k+2}|}\cdot 2k(2k-1)\right\}_{k\geqslant 1}$$

is also nondecreasing. From the well-known formula of Euler

$$\zeta(2k) = (-1)^{k+1}\frac{(2\pi)^{2k}B_{2k}}{2(2k)!} \qquad (17)$$

valid for all positive integers k, we have

$$\left\{\frac{(2k+2)|B_{2k}|}{2k|B_{2k+2}|}\cdot 2k(2k-1)\right\} = \left\{4\pi^2\frac{(2k-1)\zeta(2k)}{(2k+1)\zeta(2k+2)}\right\} \qquad (18)$$

for all $k \geqslant 1$. By means of equation (17) we verify directly that the sequence (18) is increasing for $1 \leqslant k \leqslant 3$. The fact that the sequence (18) is increasing for $k \geqslant 3$ follows from the fact that the function

$$g(s) := \frac{(s-1)\zeta(s)}{(s+1)\zeta(s+2)}$$

is increasing for $s \geqslant 6$, which is a direct consequence of the log-concavity of $(s-1)\zeta(s)$ for $s \geqslant 6$ demonstrated in Lemma 5. Therefore the sequence (18) is increasing for all $k \geqslant 1$, thus completing the proof of our Theorem 1.

5 Estimation of the roots of $\beta_n(\lambda)$

In this section, we prove Corollary 2. The given bounds are easily verified directly for $n < 4$, so in what follows we assume $n \geqslant 4$. For this, we invoke the theorem of Kakeya ([7],[4]), which says that for a polynomial $p(x) := a_0 + a_1 x + \cdots + a_m x^m$ with positive real coefficients a_i, the minimum value in

$$\frac{a_0}{a_1}, \frac{a_1}{a_2}, \ldots, \frac{a_{m-1}}{a_m}$$

is a lower bound, and the maximum such value is a upper bound, for the absolute value of a root of $p(x)$. For $p(x) = \alpha_n(x)$, we have shown that the minimum such value is

$$\frac{a_0}{a_1} = 12(n-1)|b_n|.$$

By Lemma 3, we have

$$12(n-1)|b_n| > \frac{12(n-1)}{8n(1+\log(n-1))^2} > \frac{1}{(1+\log(n-1))^2}. \tag{19}$$

Thus, every root x of $\alpha_n(x)$ satisfies $|x| \geqslant (1+\log(n-1))^{-2}$. Since a nonzero value λ is a root of $\beta_n(\lambda)$ if and only if $x = -1/\lambda^2$ is a root of $\alpha_n(x)$ (see transformation (4)), it follows that every root λ of $\beta_n(\lambda)$ satisfies $|\lambda| \leqslant 1 + \log(n-1)$, giving our stated upper bound.

By Theorem 1, if $n = 2m$ is even, the largest of our coefficient ratios is

$$\frac{a_{m-1}}{a_m} = \frac{2m|B_{2m-2}s(2m-1,2m-3)|}{(2m-2)|B_{2m}s(2m-1,2m-1)|}. \tag{20}$$

It is immediate from estimate (2) that $s(n-1,n-1)=1$, and

$$\begin{aligned} s(n-1,n-3) &= \sum_{1\leqslant i<j\leqslant n-2} ij \\ &= \frac{1}{2}\left(\left(\sum_{k=1}^{n-2} k\right)^2 - \sum_{k=1}^{n-2} k^2\right) \\ &= \frac{1}{2}\left(\left(\frac{(n-1)(n-2)}{2}\right)^2 - \frac{(n-1)(n-2)(2n-3)}{6}\right) \\ &< \frac{(n-1)^2(n-2)^2}{8}. \end{aligned} \tag{21}$$

From Euler's formula (17), we have

$$\frac{2m|B_{2m-2}|}{(2m-2)|B_{2m}|} = \frac{4\pi^2\zeta(2m-2)}{(2m-1)(2m-2)\zeta(2m)}, \tag{22}$$

so putting together (20), (21) (recall that $n=2m$), and (22), we arrive at

$$\frac{a_{m-1}}{a_m} < \frac{\pi^2}{2}(2m-1)(2m-2)\frac{\zeta(2m-2)}{\zeta(2m)}. \tag{23}$$

Since $\zeta(2m)>1$ and $(2m-2)\zeta(2m-2)\leqslant(2m-1)$ for $m>2$ by Lemma 4, we get from inequality (23), that

$$\frac{a_{m-1}}{a_m} < \frac{\pi^2(2m-1)^2}{2}.$$

Thus, by Kakeya's theorem mentioned above, every root x of $\alpha_n(x)$ satisfies $|x|\leqslant \pi^2(n-1)^2/2$, which implies, via transformation (4), that every root λ of $\beta_n(\lambda)$ satisfies $|\lambda|\geqslant\sqrt{2}/(\pi(n-1))$, when n is even.

By Theorem 1, if $n=2m+1$ is odd, the largest of our coefficient ratios is

$$\frac{a_{m-1}}{a_m} = \frac{2m|B_{2m-2}s(2m,2m-3)|}{(2m-2)|B_{2m}s(2m,2m-1)|}. \tag{24}$$

It is immediate from (2) that $|s(n-1,n-2)| = (n-1)(n-2)/2$, and

$$\begin{aligned} |s(n-1,n-4)| &= \sum_{1\leqslant i<j<k\leqslant n-2} ijk \\ &< \frac{1}{6}\left(\sum_{k=1}^{n-2} k\right)^3 \\ &= \frac{(n-1)^3(n-2)^3}{48}, \end{aligned}$$

and therefore

$$\frac{|s(n-1,n-4)|}{|s(n-1,n-2)|} < \frac{(n-1)^2(n-2)^2}{24}. \tag{25}$$

Now combining (24), (25) (recall that $n = 2m+1$), and (22), we get

$$\frac{a_{m-1}}{a_m} < \frac{\pi^2}{6}(2m-1)(2m-2)\frac{\zeta(2m-2)}{\zeta(2m)}. \tag{26}$$

Since $\zeta(2m) > 1$ and $(2m-2)\zeta(2m-2) \leqslant (2m-1)$ for $m > 2$ again by Lemma 4, we get from estimate (26) that

$$\frac{a_{m-1}}{a_m} < \frac{\pi^2(2m-1)^2}{6}.$$

Thus, every root x of $\alpha_n(x)$ satisfies $|x| \leqslant \pi^2(n-2)^2/6$, which implies, again via transformation (4), that every nonzero root λ of $\beta_n(\lambda)$ satisfies $|\lambda| \geqslant \sqrt{6}/(\pi(n-2))$, when n is odd. This completes the proof of Corollary 2.

Acknowledgements. This work started during a visit of P. T. Y. at the School of Mathematics of the University of the Witwatersrand in May of 2011. He thanks the people of this department for their hospitality and support. This paper was written in Spring of 2011 while F. L. was on sabbatical from the Mathematical Institute UNAM from January 1 to June 30, 2011 and supported by a PASPA fellowship from DGAPA.

References

[1] A. Adelberg, A finite difference approach to degenerate Bernoulli and Stirling polynomials, *Discrete Math.* **140** (1995), 1-21.

[2] T. M. Apostol, *Introduction to Analytic Number Theory*, Springer–Verlag New York, 1976.

[3] L. Carlitz, Degenerate Stirling, Bernoulli, and Eulerian numbers, *Utilitas Math.* **15** (1979), 51-88.

[4] K. Dilcher, Zeros of Bernoulli, generalized Bernoulli, and Euler polynomials, *Mem. Amer. Math. Soc.* **73** (1988), no. 386, iv+94pp.

[5] G. Hetyei, Enumeration by kernel positions, *Adv. Appl. Math.* **42** (2009), 445-470.

[6] F. T. Howard, Explicit formulas for degenerate Bernoulli numbers, *Discrete Math.* **162** (1996), 175-185.

[7] S. Kakeya, On the limits of the roots of an algebraic equation with positive coefficients, *Tohoku Math. J.* **2** (1912), 140-142.

[8] M. Sibuya, Log-concavity of Stirling numbers and unimodality of Stirling distributions, *Ann. Inst. Stat. Math.* **40.4** (1988), 693-714.

[9] P. T. Young, Degenerate Bernoulli polynomials, generalized factorial sums, and their applications, *J. Number Theory* **128.4** (2008), 738-758.

[10] P. T. Young, Bernoulli numbers and generalized factorial sums, *INTEGERS* **11.4** (2011), 553-561.