ON THE LOG-CONCAVITY OF THE DEGENERATE BERNOULLI NUMBERS

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Abstract
The degenerate Bernoulli numbers $\beta_n(\lambda)$ are polynomials with rational coefficients of degree $n$ in the variable $\lambda$, which arise in several combinatorial settings. An appropriate change of variable transforms $\beta_n(\lambda)$ into a polynomial whose coefficients are all positive. Here, we prove that this transformed polynomial is log-concave, and therefore unimodal. As a consequence, we deduce bounds on the absolute values of the roots of $\beta_n(\lambda)$. 
1 Introduction

Carlitz [3] defined the degenerate Bernoulli numbers $\beta_n(\lambda)$ for $\lambda \neq 0$ by means of the generating function

$$\frac{t}{(1+\lambda t)^\mu - 1} = \sum_{n=0}^{\infty} \beta_n(\lambda) \frac{t^n}{n!},$$

where $\lambda \mu = 1$. Each $\beta_n(\lambda)$ is a polynomial of degree $n$ in $\lambda$ with rational coefficients; for $n > 1$ the polynomial $\beta_n(\lambda)$ is an even (resp. odd) function of $\lambda$ when $n$ is even (resp. odd). These polynomials occur in expressions for sums of falling factorials [3, 9], for divided differences of binomial coefficients [1], for coset products in factor rings [10], and in game theory probabilities [5]. Since $(1+\lambda t)^\mu \to e^t$ as $\lambda \to 0$, it is evident that $\beta_n(0) = B_n$, the $n$-th Bernoulli number, which is defined by

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!};$$

since $((1+t)^\mu - 1)/\mu \to \log(1+t)$ as $\mu \to 0$, we have $\lim_{\lambda \to \infty} \lambda^{-n} \beta_n(\lambda) = n! b_n$, where the Bernoulli numbers of the second kind $b_n$ are defined by

$$\frac{t}{\log(1+t)} = \sum_{n=0}^{\infty} b_n t^n.$$

Thus, $\beta_n(\lambda)$ is a polynomial of degree $n$ in $\lambda$ whose constant term is $B_n$ and whose leading coefficient is $n! b_n$.

For $n > 1$, Howard [6] gave an explicit formula for the coefficients of $\beta_n(\lambda)$,

$$\beta_n(\lambda) = n! b_n \lambda^n + \sum_{k=1}^{\lfloor n/2 \rfloor} \frac{n B_{2k}}{2^k} s(n - 1, 2k - 1) \lambda^{n-2k}, \quad (1)$$

involving the Stirling numbers $s(n, k)$ of the first kind, which may be defined by

$$x(x-1) \cdots (x-n+1) = \sum_{k=0}^{n} s(n, k) x^k. \quad (2)$$

The first few values of $\beta_n(\lambda)$ are

$$\beta_0(\lambda) = 1, \quad \beta_1(\lambda) = -\frac{1}{2} + \frac{1}{2} \lambda, \quad \beta_2(\lambda) = \frac{1}{6} - \frac{1}{6} \lambda^2,$$
\[\beta_3(\lambda) = -\frac{1}{4} \lambda + \frac{1}{4} \lambda^3, \quad \beta_4(\lambda) = -\frac{1}{30} + \frac{2}{3} \lambda^2 - \frac{19}{30} \lambda^4,\]

\[\beta_5(\lambda) = \frac{1}{4} \lambda - \frac{5}{2} \lambda^3 + \frac{9}{4} \lambda^5, \quad \beta_6(\lambda) = \frac{1}{42} - \frac{7}{4} \lambda^2 + 12 \lambda^4 - \frac{863}{84} \lambda^6,\]

\[\beta_7(\lambda) = -\frac{1}{12} \lambda + \frac{105}{8} \lambda^3 - 70 \lambda^5 + \frac{1375}{24} \lambda^7,\]

\[\beta_8(\lambda) = -\frac{1}{30} + \frac{50}{9} \lambda^2 - \frac{1624}{15} \lambda^4 + 480 \lambda^6 - \frac{33953}{90} \lambda^8,\]

\[\beta_9(\lambda) = \frac{21}{20} \lambda - 70 \lambda^3 + \frac{9849}{10} \lambda^5 - 3780 \lambda^7 + \frac{57281}{20} \lambda^9.\]

It is easy to verify that for even \(n\) the polynomial \(\beta_n(\lambda)\) given by (1) is a polynomial in \(\lambda^2\) whose nonzero coefficients alternate in sign, as is \(\beta_n(\lambda)/\lambda\) if \(n > 1\) is odd. Here, we consider the transformed polynomial \(\alpha_n(x)\) defined for \(n > 1\) by

\[\alpha_n(x) := n!|b_n| + \sum_{k=1}^{[n/2]} \frac{n!|B_{2k}|}{2^k} s(n - 1, 2k - 1)x^k, \quad (3)\]

which is a polynomial of degree \([n/2]\) in \(x\) having all coefficients positive, in agreement up to sign, in reverse order, with the nonzero coefficients of \(\beta_n(\lambda)\).

The polynomials shown at NYO and N\[O\] are related by the change of variables

\[\beta_n(\lambda) = -(-\lambda)^n \alpha_n(-1/\lambda^2), \quad \alpha_n(x) = -(-\sqrt{-x})^{n} \beta_n(\sqrt{-1/x}). \quad (4)\]

A polynomial \(p(x) = a_0 + a_1x + \cdots + a_mx^m\) with positive real coefficients \(a_i\) is called log-concave if its sequence of coefficients \(\{a_0, a_1, \ldots, a_m\}\) is logarithmically concave, meaning that \(a_{i-1}a_{i+1} \leq a_i^2\) for \(0 < i < m\). The main result of this paper is the following:

**Theorem 1.** For all \(n > 1\), the polynomial \(\alpha_n(x)\) is log-concave.

It is well-known that any log-concave polynomial is unimodal, meaning that there exists an index \(j\) for which \(a_0 \leq a_1 \leq \cdots \leq a_j \geq a_{j+1} \geq \cdots \geq a_m\). We use the Theorem 1 to deduce the following bounds for the roots of \(\beta_n(\lambda)\).

**Corollary 2.** For all even \(n > 1\), if \(\lambda \in \mathbb{C}\) is a root of the polynomial \(\beta_n(\lambda)\), then

\[\frac{\sqrt{2}}{\pi(n - 1)} \leq |\lambda| \leq 1 + \log(n - 1),\]
and for all odd \( n > 1 \), if \( \lambda \in \mathbb{C} \) is a nonzero root of the polynomial \( \beta_n(\lambda) \), then

\[
\frac{\sqrt{6}}{\pi(n-2)} \leq |\lambda| \leq 1 + \log(n-1).
\]

It is well known that \( \lambda = \pm 1 \) is a root of \( \beta_n(\lambda) \) for all \( n > 1 \), and it was recently shown in [9], that if \( n \) is odd, then \( \lambda = \pm 1/d \) is a root of \( \beta_n(\lambda) \) for every divisor \( d \) of \( n - 2 \); so, in particular, for odd \( n \) we have rational roots at \( \lambda = \pm 1/(n - 2) \) and \( \lambda = \pm 1 \). A primary motivation for this paper was to examine the size of the other roots of \( \beta_n(\lambda) \). Based on numerical computations, the lower bounds given in Corollary 2 appear to be fairly sharp. On the other hand, for \( n \leq 100 \) the roots of \( \beta_n(\lambda) \) all satisfy \( |\lambda| \leq 1 \), which is quite a bit smaller than our stated upper bound. We conjecture that all roots \( \lambda \) of all the polynomials \( \beta_n(\lambda) \) have absolute values \( |\lambda| \leq 1 \).

## 2 Estimates for the Bernoulli numbers of the second kind

In order to prove our results, we need some estimates on the growth of the sequence \( \{b_n\}_{n \geq 1} \). It is well-known [9, 10], that the Bernoulli numbers of the second kind \( b_n \) may also be defined by

\[
n!b_n = \int_0^1 x(x-1) \cdots (x-n+1) dx.
\]

**Lemma 3.** We have

\[
\frac{1}{8n(1 + \log(n-1))^2} < |b_n| < \frac{1}{n(\log n)^2}
\]

for all \( n \geq 2 \).

**Proof.** The inequalities may be verified directly for \( n = 2, 3, 4 \). Considering tangent lines and secant lines for \( f(t) = e^{-t} \), we have the inequalities

\[
1 - t \leq e^{-t} \leq 1 - rt \quad \text{for} \quad t \in [0, 1],
\]

where \( r := 1 - e^{-1} \). Considering the upper and lower approximations by rectangles to the integral of \( g(t) = 1/t \) on the interval \( [1, n] \), we have

\[
\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} < \log n < 1 + \frac{1}{2} + \cdots + \frac{1}{n-1}.
\]
By the integral formula (5) for $b_n$ and the left inequality in (7), we have

\[
n!|b_n| = \int_0^1 x(1-x)(2-x) \cdots (n-1-x)dx \\
= (n-1)! \int_0^1 x \left(1 - \frac{x}{1}\right) \left(1 - \frac{x}{2}\right) \cdots \left(1 - \frac{x}{n-1}\right) dx \\
< (n-1)! \int_0^1 x \exp \left(-x \left(1 + \frac{1}{2} + \cdots + \frac{1}{n-1}\right)\right) dx. \quad (9)
\]

Then, by the left inequality in (8), we have

\[
-x \left(1 + \frac{1}{2} + \cdots + \frac{1}{n-1}\right) < -x \log n \quad \text{for all} \quad x \in (0,1]. \quad (10)
\]

Inserting estimate (10) into (9), we get

\[
n!|b_n| < (n-1)! \int_0^1 x \exp(-x \log n) dx.
\]

With the change of variable $u := x \log n$ in the above integral, for which $du = (\log n)dx$, we have that

\[
n!|b_n| < \frac{(n-1)!}{(\log n)^2} \int_0^{\log n} u e^{-u} du = \frac{(n-1)!}{(\log n)^2} \left(-ue^{-u} - e^{-u} \bigg|_{u=\log n}^{u=0}\right) \\
= \frac{(n-1)!}{(\log n)^2} \left(1 - \frac{1 + \log n}{n}\right) < \frac{(n-1)!}{(\log n)^2},
\]

which implies the right inequality in (6).

Again by the integral formula (5) and the right inequality in (7), we deduce, with $x := rt$, that

\[
n!|b_n| = \int_0^1 x(1-x)(2-x) \cdots (n-1-x)dx \\
= (n-1)! \int_0^1 x \left(1 - \frac{x}{1}\right) \left(1 - \frac{x}{2}\right) \cdots \left(1 - \frac{x}{n-1}\right) dx \\
= (n-1)! \int_0^{r^{-1}} r^2 t \left(1 - \frac{rt}{1}\right) \left(1 - \frac{rt}{2}\right) \cdots \left(1 - \frac{rt}{n-1}\right) dt \\
> (n-1)! \int_0^{r^{-1}} r^2 t \exp \left(-t \left(1 + \frac{1}{2} + \cdots + \frac{1}{n-1}\right)\right) dt. \quad (11)
\]
Then, by the right inequality in (8), we have
\[-x \left(1 + \frac{1}{2} + \cdots + \frac{1}{n-1}\right) > -x(1 + \log(n-1)) \quad \text{for all} \quad x \in (0,1]. \quad (12)\]

Inserting estimate (12) into (11), we get
\[n!|b_n| > (n - 1)! \int_0^{r^{-1}} r^2 t \exp(-t(1 + \log(n - 1))) dt.\]

Integration by parts above then yields
\[n|b_n| > \left( -r e^{-(1+\log(n-1))} - \frac{r^2 e^{-(1+\log(n-1))}}{(1+\log(n-1))^2} + \frac{r^2}{(1+\log(n-1))^2} \right) \frac{r^2}{(1+L)^2} \left( 1 - \frac{e^{-1}}{n-1} \left( 1 + \frac{1+L}{r} \right) \right),\]

where $L := \log(n - 1)$.

Since it is easily established that $(1 + L)/(n - 1) \leq 1$ for all $n$ (in fact, this follows from the left inequality (7) with $t := -\log(n - 1)$), we have
\[n|b_n| > \frac{r^2}{(1+L)^2} \left( 1 - \frac{e^{-1}}{n-1} - \frac{e^{-1}}{1-e^{-1}} \right).\]

Therefore, when $n > 4$, we have
\[n|b_n| > \frac{1}{8(1+L)^2},\]
giving the left inequality of (6) and completing the proof. \qed

3 The log-concavity of $(s - 1)\zeta(s)$

To prove our Theorem 1, we need to verify that $(s - 1)\zeta(s)$ is log-concave for sufficiently large real numbers $s > 1$, where $\zeta(s)$ is the Riemann zeta function. We begin with the following elementary estimates.

Lemma 4. We have $s\zeta(s) \leq s + 1$ for $s \geq 3$, and $s^2\zeta(s) \leq s^2 + 1$ for $s \geq 5$. 

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Proof. Using the series for $\zeta(s)$ as an approximation to an integral with rectangles from below, we get

$$
\zeta(s) = 1 + 2^{-s} + 3^{-s} + \cdots
< 1 + 2^{-s} + \int_2^\infty x^{-s} \, dx
= 1 + 2^{-s} + \frac{2^{1-s}}{s-1}
\leq 1 + \frac{1}{s} \text{ for } s \geq 3,
$$
giving the first statement. Since

$$
1 + 2^{-s} + \frac{2^{1-s}}{s-1} \leq 1 + \frac{1}{s^2} \text{ for } s \geq 5,
$$
the second statement follows as well.

\[\]

Let

$$f(s) := \log((s-1)\zeta(s)) \text{ for } s > 1.$$ 

Lemma 5. We have $f''(s) < 0$ for all $s \geq 6$.

Proof. We first observe that

$$
f'(s) = \frac{d}{ds} (\log(s-1) + \log \zeta(s)) = \frac{1}{s-1} + \frac{\zeta'(s)}{\zeta(s)}
= \frac{1}{s-1} - \sum_{n=2}^\infty \Lambda(n)n^{-s},
$$
where

$$\Lambda(n) = \begin{cases} 
\log p, & \text{if } n = p^m \text{ for some prime } p \text{ and integer } m \geq 1, \\
0, & \text{otherwise}
\end{cases}$$
is the von–Mangoldt function, and therefore

$$f''(s) = -\frac{1}{(s-1)^2} + \sum_{n=2}^\infty \Lambda(n)(\log n)n^{-s}$$

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for all $s > 1$. Since $0 \leq \Lambda(n) \leq \log n < n^{1/2}$ for $n \geq 1$, we have by Lemma 4

$$f''(s) \leq -\frac{1}{(s-1)^2} + \sum_{n=2}^{\infty} \Lambda(n)(\log n)n^{-s}$$

$$< -\frac{1}{(s-1)^2} + \sum_{n=2}^{\infty} n \cdot n^{-s} \quad \text{(for } s > 2)$$

$$= -\frac{1}{(s-1)^2} + (\zeta(s-1) - 1)$$

$$\leq -\frac{1}{(s-1)^2} + \frac{1}{(s-1)^2} = 0 \quad \text{for } s \geq 6,$$

as desired. \hfill \Box

**Remark.** While not needed for our purposes, we can verify numerically that in fact $f''(s) < 0$ for all $s > 1$. To estimate $f''(s)$ for $s > 1$ and near $1$, we use the formula ([2], p. 56)

$$\zeta(s) = \frac{s}{s-1} - s \int_{1}^{\infty} \frac{\{x\}}{x^{s+1}}dx := \frac{s}{s-1} - sg(s),$$

where $\{x\} = x - [x]$ denotes the fractional part of $x$ and

$$g(s) := \int_{1}^{\infty} \frac{\{x\}}{x^{s+1}}dx.$$}

Then

$$f(s) = \log((s-1)\zeta(s)) = \log(s - s(s-1)g(s)) = \log s + \log(1 - (s-1)g(s)).$$

For $1 < s < 2$, we have that

$$0 < (s-1)g(s) = (s-1) \int_{1}^{\infty} \frac{\{x\}}{x^{s+1}}dx < (s-1) \int_{1}^{\infty} \frac{dx}{x^2} = s - 1 < 1,$$

so we can expand the logarithm in series getting

$$f(s) = \log s - \sum_{k=1}^{\infty} \frac{1}{k} (s-1)^k g(s)^k = \log s - (s-1)g(s) - \frac{1}{2}(s-1)^2 g(s)^2 + (s-1)^3 h(s),$$

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where $h(s)$ is analytic in a neighborhood of $1$. Taking derivatives we get

$$f'(s) = \frac{1}{s} - g(s) - (s-1)g'(s) - (s-1)g(s)^2 + (s-1)^2a(s),$$

where $a(s) := -g(s)g'(s) + 3h(s) + (s-1)h'(s)$. So,

$$f''(s) = -\frac{1}{s^2} - 2g'(s) - g(s)^2 + (s-1)b(s),$$

where $b(s) := -g''(s) - 2g(s)g'(s) + 2a(s) + (s-1)a'(s)$. Thus, $f''(s)$ in a neighborhood of $1$ is close to

$$-1 - 2g'(1) - g(1)^2 = -1 + 2\int_{1}^{\infty} \frac{\{x\} \log x}{x^2} dx - \left( \int_{1}^{\infty} \frac{\{x\} \log x}{x^2} dx \right)^2,$$

a number which is about $-0.19$.

## 4 The log-concavity of $\alpha_n(x)$

In this section, we give the proof of Theorem 1. Log-concavity of $\alpha_n(x)$ is easily verified directly for $n \leq 4$, so in what follows we assume $n \geq 4$. The polynomial $\alpha_n(x) = a_0 + a_1x + \cdots + a_mx^m$ has degree $m := \lfloor n/2 \rfloor$, with coefficients $a_0 := n!|b_n|$ and $a_k := (n|B_{2k}|/2k)|s(n-1,2k-1)|$ for $1 \leq k \leq m$. Log-concavity of $\alpha_n(x)$ is equivalent to the statement that the sequence of ratios of coefficients

$$\frac{a_0}{a_1}, \frac{a_1}{a_2}, \ldots, \frac{a_{m-1}}{a_m}$$

is nondecreasing.

For $\alpha_n(x)$ we have, by Lemma 3, that

$$\frac{a_0}{a_1} = \frac{n!|b_n|}{(n/2)|B_{2s(n-1,1)|} = 12(n-1)|b_n| < \frac{12}{(\log n)^2}. \quad (14)$$

By direct calculation, we have

$$\frac{a_1}{a_2} = \frac{4|B_{2s(n-1,1)}|}{2|B_{4s(n-1,3)}|} = 10 \frac{|s(n-1,1)|}{|s(n-1,3)|}. \quad (15)$$

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From equation (4.8) in [9], we have
\[
|s(n - 1, 3)| = |s(n - 1, 1)| \sum_{1 \leq i < j \leq n-2} \frac{1}{i j} \]
\[
= \frac{|s(n - 1, 1)|}{2} \left( \left( \sum_{k=1}^{n-2} \frac{1}{k} \right)^2 - \sum_{k=1}^{n-2} \frac{1}{k^2} \right) \]
\[
< \frac{|s(n - 1, 1)|}{2} \left( (1 + \log(n - 2))^2 - 1 \right), \tag{16}
\]
where the last inequality above follows by using the left inequality of (8). Therefore, by estimates (14), (15), and (16), we have
\[
\frac{a_0}{a_1} < \frac{12}{(\log n)^2} < \frac{20}{(1 + \log(n - 2))^2 - 1} < \frac{a_1}{a_2},
\]
thus demonstrating the first inequality required in (13).

The log-concavity of \(\alpha_n(x)\) will then be demonstrated by showing that the sequence
\[
\left\{ \frac{a_k}{a_{k+1}} \right\} = \left\{ \frac{(2k + 2)|B_{2k} s(n - 1, 2k - 1)|}{2k|B_{2k+2} s(n - 1, 2k + 1)|} \right\}
\]
for \(1 \leq k < m\) is nondecreasing. From Theorem 3.1 in [8], we know that the sequence
\[
\left\{ \frac{1}{2k(2k - 1)} \left| s(n - 1, 2k - 1) \right| \over \left| s(n - 1, 2k + 1) \right| \right\}_{k \geq 1}
\]
is an increasing sequence in \(k\). It therefore suffices to show that the sequence
\[
\left\{ \frac{(2k + 2)|B_{2k}|}{2k|B_{2k+2}|} \cdot 2k(2k - 1) \right\}_{k \geq 1}
\]
is also nondecreasing. From the well-known formula of Euler
\[
\zeta(2k) = (-1)^{k+1} \frac{(2\pi)^{2k} B_{2k}}{2(2k)!} \tag{17}
\]
valid for all positive integers \(k\), we have
\[
\left\{ \frac{(2k + 2)|B_{2k}|}{2k|B_{2k+2}|} \cdot 2k(2k - 1) \right\} = \left\{ 4\pi^2 \frac{(2k - 1)\zeta(2k)}{(2k + 1)\zeta(2k + 2)} \right\} \tag{18}
\]
for all $k \geq 1$. By means of equation (17) we verify directly that the sequence (18) is increasing for $1 \leq k \leq 3$. The fact that the sequence (18) is increasing for $k \geq 3$ follows from the fact that the function

$$g(s) := \frac{(s-1) \zeta(s)}{(s+1) \zeta(s+2)}$$

is increasing for $s \geq 6$, which is a direct consequence of the log-concavity of $(s-1) \zeta(s)$ for $s \geq 6$ demonstrated in Lemma 5. Therefore the sequence (18) is increasing for all $k \geq 1$, thus completing the proof of our Theorem 1.

5 Estimation of the roots of $\beta_n(\lambda)$

In this section, we prove Corollary 2. The given bounds are easily verified directly for $n < 4$, so in what follows we assume $n \geq 4$. For this, we invoke the theorem of Kakeya ([7],[4]), which says that for a polynomial $p(x) := a_0 + a_1 x + \cdots + a_m x^m$ with positive real coefficients $a_i$, the minimum value in

$$\frac{a_0}{a_1}, \frac{a_1}{a_2}, \ldots, \frac{a_{m-1}}{a_m}$$

is a lower bound, and the maximum such value is an upper bound, for the absolute value of a root of $p(x)$. For $p(x) = \alpha_n(x)$, we have shown that the minimum such value is

$$\frac{a_0}{a_1} = 12(n-1)|b_n|.$$ 

By Lemma 3, we have

$$12(n-1)|b_n| > \frac{12(n-1)}{8n(1 + \log(n-1))^2} > \frac{1}{(1 + \log(n-1))^2}. \quad (19)$$

Thus, every root $x$ of $\alpha_n(x)$ satisfies $|x| \geq (1 + \log(n-1))^{-2}$. Since a nonzero value $\lambda$ is a root of $\beta_n(\lambda)$ if and only if $x = -1/\lambda^2$ is a root of $\alpha_n(x)$ (see transformation (4)), it follows that every root $\lambda$ of $\beta_n(\lambda)$ satisfies $|\lambda| \leq 1 + \log(n-1)$, giving our stated upper bound.

By Theorem 1, if $n = 2m$ is even, the largest of our coefficient ratios is

$$\frac{a_{m-1}}{a_m} = \frac{2m|B_{2m-2}s(2m-1,2m-3)|}{(2m-2)|B_{2m}s(2m-1,2m-1)|}. \quad (20)$$
It is immediate from estimate (2) that \(s(n - 1, n - 1) = 1\), and

\[
s(n - 1, n - 3) = \sum_{1 \leq i < j \leq n-2} ij
\]
\[
= \frac{1}{2} \left( \left( \sum_{k=1}^{n-2} k \right)^2 - \sum_{k=1}^{n-2} k^2 \right)
\]
\[
= \frac{1}{2} \left( \left( \frac{(n-1)(n-2)}{2} \right)^2 - \frac{(n-1)(n-2)(2n-3)}{6} \right)
\]
\[
< \frac{(n-1)^2(n-2)^2}{8}. \tag{21}
\]

From Euler’s formula (17), we have

\[
\frac{2m|B_{2m-2}|}{(2m-2)|B_{2m}|} = \frac{4\pi^2\zeta(2m-2)}{(2m-1)(2m-2)\zeta(2m)}, \tag{22}
\]

so putting together (20), (21) (recall that \(n = 2m\)), and (22), we arrive at

\[
\frac{a_{m-1}}{a_m} < \frac{\pi^2(2m-1)(2m-2)\zeta(2m-2)}{\zeta(2m)}. \tag{23}
\]

Since \(\zeta(2m) > 1\) and \((2m-2)\zeta(2m-2) \leq (2m-1)\) for \(m > 2\) by Lemma 4, we get from inequality (23), that

\[
\frac{a_{m-1}}{a_m} < \frac{\pi^2(2m-1)^2}{2}.
\]

Thus, by Kakeya’s theorem mentioned above, every root \(x\) of \(\alpha_n(x)\) satisfies \(|x| \leq \pi^2(n - 1)^2/2\), which implies, via transformation (4), that every root \(\lambda\) of \(\beta_n(\lambda)\) satisfies \(|\lambda| \geq \sqrt{2}/(\pi(n - 1))\), when \(n\) is even.

By Theorem 1, if \(n = 2m + 1\) is odd, the largest of our coefficient ratios is

\[
\frac{a_{m-1}}{a_m} = \frac{2m|B_{2m-2}s(2m, 2m-3)|}{(2m-2)|B_{2m}s(2m, 2m-1)|}. \tag{24}
\]
It is immediate from (2) that $|s(n-1,n-2)| = (n-1)(n-2)/2$, and

$$|s(n-1,n-4)| = \sum_{1\leq i<j<k\leq n-2} ijk < \frac{1}{6} \left( \sum_{k=1}^{n-2} k \right)^3 = \frac{(n-1)^3(n-2)^3}{48},$$

and therefore

$$\frac{|s(n-1,n-4)|}{|s(n-1,n-2)|} < \frac{(n-1)^2(n-2)^2}{24}. \quad (25)$$

Now combining (24), (25) (recall that $n = 2m + 1$), and (22), we get

$$\frac{a_{m-1}}{a_m} < \frac{\pi^2}{6}(2m-1)(2m-2)\frac{\zeta(2m-2)}{\zeta(2m)}. \quad (26)$$

Since $\zeta(2m) > 1$ and $(2m-2)\zeta(2m-2) \leq (2m-1)$ for $m > 2$ again by Lemma 4, we get from estimate (26) that

$$\frac{a_{m-1}}{a_m} < \frac{\pi^2(2m-1)^2}{6}.$$ 

Thus, every root $x$ of $\alpha_n(x)$ satisfies $|x| \leq \pi^2(n-2)^2/6$, which implies, again via transformation (4), that every nonzero root $\lambda$ of $\beta_n(\lambda)$ satisfies $|\lambda| \geq \sqrt{6}/(\pi(n-2))$, when $n$ is odd. This completes the proof of Corollary 2.

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References


