On the log-concavity of the degenerate Bernoulli numbers

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Abstract

The degenerate Bernoulli numbers $\beta_n(\lambda)$ are polynomials with rational coefficients of degree n in the variable λ, which arise in several combinatorial settings. An appropriate change of variable transforms $\beta_n(\lambda)$ into a polynomial whose coefficients are all positive. Here, we prove that this transformed polynomial is log-concave, and therefore unimodal. As a consequence, we deduce bounds on the absolute values of the roots of $\beta_n(\lambda)$.

1 Introduction

Carlitz [3] defined the *degenerate Bernoulli numbers* $\beta_n(\lambda)$ for $\lambda \neq 0$ by means of the generating function

$$\frac{t}{(1+\lambda t)^{\mu}-1} = \sum_{n=0}^{\infty} \beta_n(\lambda)\frac{t^n}{n!},$$

where $\lambda\mu = 1$. Each $\beta_n(\lambda)$ is a polynomial of degree n in λ with rational coefficients; for $n > 1$ the polynomial $\beta_n(\lambda)$ is an even (resp. odd) function of λ when n is even (resp. odd). These polynomials occur in expressions for sums of falling factorials [3, 9], for divided differences of binomial coefficients [1], for coset products in factor rings [10], and in game theory probabilities [5]. Since $(1+\lambda t)^{\mu} \to e^t$ as $\lambda \to 0$, it is evident that $\beta_n(0) = B_n$, the n-th *Bernoulli number*, which is defined by

$$\frac{t}{e^t-1} = \sum_{n=0}^{\infty} B_n\frac{t^n}{n!};$$

since $((1+t)^{\mu}-1)/\mu \to \log(1+t)$ as $\mu \to 0$, we have $\lim_{\lambda\to\infty} \lambda^{-n}\beta_n(\lambda) = n!\, b_n$, where the *Bernoulli numbers of the second kind* b_n are defined by

$$\frac{t}{\log(1+t)} = \sum_{n=0}^{\infty} b_n t^n.$$

Thus, $\beta_n(\lambda)$ is a polynomial of degree n in λ whose constant term is B_n and whose leading coefficient is $n!\, b_n$.

For $n > 1$, Howard [6] gave an explicit formula for the coefficients of $\beta_n(\lambda)$,

$$\beta_n(\lambda) = n! b_n \lambda^n + \sum_{k=1}^{[n/2]} \frac{nB_{2k}}{2k} s(n-1, 2k-1)\lambda^{n-2k}, \tag{1}$$

involving the *Stirling numbers* $s(n,k)$ *of the first kind*, which may be defined by

$$x(x-1)\cdots(x-n+1) = \sum_{k=0}^{n} s(n,k)x^k. \tag{2}$$

The first few values of $\beta_n(\lambda)$ are

$$\beta_0(\lambda) = 1, \qquad \beta_1(\lambda) = -\frac{1}{2} + \frac{1}{2}\lambda, \qquad \beta_2(\lambda) = \frac{1}{6} - \frac{1}{6}\lambda^2,$$

$$\beta_3(\lambda) = -\frac{1}{4}\lambda + \frac{1}{4}\lambda^3, \qquad \beta_4(\lambda) = -\frac{1}{30} + \frac{2}{3}\lambda^2 - \frac{19}{30}\lambda^4,$$
$$\beta_5(\lambda) = \frac{1}{4}\lambda - \frac{5}{2}\lambda^3 + \frac{9}{4}\lambda^5, \qquad \beta_6(\lambda) = \frac{1}{42} - \frac{7}{4}\lambda^2 + 12\lambda^4 - \frac{863}{84}\lambda^6,$$
$$\beta_7(\lambda) = -\frac{5}{12}\lambda + \frac{105}{8}\lambda^3 - 70\lambda^5 + \frac{1375}{24}\lambda^7,$$
$$\beta_8(\lambda) = -\frac{1}{30} + \frac{50}{9}\lambda^2 - \frac{1624}{15}\lambda^4 + 480\lambda^6 - \frac{33953}{90}\lambda^8,$$
$$\beta_9(\lambda) = \frac{21}{20}\lambda - 70\lambda^3 + \frac{9849}{10}\lambda^5 - 3780\lambda^7 + \frac{57281}{20}\lambda^9.$$

It is easy to verify that for even n the polynomial $\beta_n(\lambda)$ given by (1) is a polynomial in λ^2 whose nonzero coefficients alternate in sign, as is $\beta_n(\lambda)/\lambda$ if $n > 1$ is odd. Here, we consider the transformed polynomial $\alpha_n(x)$ defined for $n > 1$ by

$$\alpha_n(x) := n!|b_n| + \sum_{k=1}^{[n/2]} \frac{n|B_{2k}|}{2k} |s(n-1, 2k-1)| x^k, \tag{3}$$

which is a polynomial of degree $[n/2]$ in x having all coefficients positive, in agreement up to sign, in reverse order, with the nonzero coefficients of $\beta_n(\lambda)$. The polynomials shown at (1) and (3) are related by the change of variables

$$\beta_n(\lambda) = -(-\lambda)^n \alpha_n(-1/\lambda^2), \quad \alpha_n(x) = -\left(-\sqrt{-x}\right)^n \beta_n(\sqrt{-1/x}). \tag{4}$$

A polynomial $p(x) = a_0 + a_1 x + \cdots + a_m x^m$ with positive real coefficients a_i is called *log-concave* if its sequence of coefficients $\{a_0, a_1, ..., a_m\}$ is logarithmically concave, meaning that $a_{i-1}a_{i+1} \leqslant a_i^2$ for $0 < i < m$. The main result of this paper is the following:

Theorem 1. *For all $n > 1$, the polynomial $\alpha_n(x)$ is log-concave.*

It is well-known that any log-concave polynomial is *unimodal*, meaning that there exists an index j for which $a_0 \leqslant a_1 \leqslant \cdots \leqslant a_j \geqslant a_{j+1} \geqslant \cdots \geqslant a_m$. We use the Theorem 1 to deduce the following bounds for the roots of $\beta_n(\lambda)$.

Corollary 2. *For all even $n > 1$, if $\lambda \in \mathbb{C}$ is a root of the polynomial $\beta_n(\lambda)$, then*

$$\frac{\sqrt{2}}{\pi(n-1)} \leqslant |\lambda| \leqslant 1 + \log(n-1),$$

and for all odd $n > 1$, if $\lambda \in \mathbb{C}$ is a nonzero root of the polynomial $\beta_n(\lambda)$, then

$$\frac{\sqrt{6}}{\pi(n-2)} \leqslant |\lambda| \leqslant 1 + \log(n-1).$$

It is well known that $\lambda = \pm 1$ is a root of $\beta_n(\lambda)$ for all $n > 1$, and it was recently shown in [9], that if n is odd, then $\lambda = \pm 1/d$ is a root of $\beta_n(\lambda)$ for every divisor d of $n-2$; so, in particular, for odd n we have rational roots at $\lambda = \pm 1/(n-2)$ and $\lambda = \pm 1$. A primary motivation for this paper was to examine the size of the other roots of $\beta_n(\lambda)$. Based on numerical computations, the lower bounds given in Corollary 2 appear to be fairly sharp. On the other hand, for $n \leqslant 100$ the roots of $\beta_n(\lambda)$ all satisfy $|\lambda| \leqslant 1$, which is quite a bit smaller than our stated upper bound. We conjecture that all roots λ of all the polynomials $\beta_n(\lambda)$ have absolute values $|\lambda| \leqslant 1$.

2 Estimates for the Bernoulli numbers of the second kind

In order to prove our results, we need some estimates on the growth of the sequence $\{b_n\}_{n\geqslant 1}$. It is well-known [9, 10], that the Bernoulli numbers of the second kind b_n may also be defined by

$$n!b_n = \int_0^1 x(x-1)\cdots(x-n+1)dx. \tag{5}$$

Lemma 3. *We have*

$$\frac{1}{8n(1+\log(n-1))^2} < |b_n| < \frac{1}{n(\log n)^2} \tag{6}$$

for all $n \geqslant 2$.

Proof. The inequalities may be verified directly for $n = 2, 3, 4$. Considering tangent lines and secant lines for $f(t) = e^{-t}$, we have the inequalities

$$1 - t \leqslant e^{-t} \leqslant 1 - rt \quad \text{for} \quad t \in [0,1], \tag{7}$$

where $r := 1 - e^{-1}$. Considering the upper and lower approximations by rectangles to the integral of $g(t) = 1/t$ on the interval $[1, n]$, we have

$$\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} < \log n < 1 + \frac{1}{2} + \cdots + \frac{1}{n-1}. \tag{8}$$

By the integral formula (5) for b_n and the left inequality in (7), we have

$$\begin{aligned} n!|b_n| &= \int_0^1 x(1-x)(2-x)\cdots(n-1-x)dx \\ &= (n-1)!\int_0^1 x\left(1-\frac{x}{1}\right)\left(1-\frac{x}{2}\right)\cdots\left(1-\frac{x}{n-1}\right)dx \\ &< (n-1)!\int_0^1 x\exp\left(-x\left(1+\frac{1}{2}+\cdots+\frac{1}{n-1}\right)\right)dx. \qquad (9) \end{aligned}$$

Then, by the left inequality in (8), we have

$$-x\left(1+\frac{1}{2}+\cdots+\frac{1}{n-1}\right) < -x\log n \quad \text{for all} \quad x\in(0,1]. \qquad (10)$$

Inserting estimate (10) into (9), we get

$$n!|b_n| < (n-1)!\int_0^1 x\exp(-x\log n)dx.$$

With the change of variable $u := x\log n$ in the above integral, for which $du = (\log n)dx$, we have that

$$\begin{aligned} n!|b_n| &< \frac{(n-1)!}{(\log n)^2}\int_0^{\log n} ue^{-u}du = \frac{(n-1)!}{(\log n)^2}\left(-ue^{-u}-e^{-u}\Big|_{u=0}^{u=\log n}\right) \\ &= \frac{(n-1)!}{(\log n)^2}\left(1-\frac{1+\log n}{n}\right) < \frac{(n-1)!}{(\log n)^2}, \end{aligned}$$

which implies the right inequality in (6).

Again by the integral formula (5) and the right inequality in (7), we deduce, with $x =: rt$, that

$$\begin{aligned} n!|b_n| &= \int_0^1 x(1-x)(2-x)\cdots(n-1-x)dx \\ &= (n-1)!\int_0^1 x\left(1-\frac{x}{1}\right)\left(1-\frac{x}{2}\right)\cdots\left(1-\frac{x}{n-1}\right)dx \\ &= (n-1)!\int_0^{r^{-1}} r^2t\left(1-\frac{rt}{1}\right)\left(1-\frac{rt}{2}\right)\cdots\left(1-\frac{rt}{n-1}\right)dt \\ &> (n-1)!\int_0^{r^{-1}} r^2t\exp\left(-t\left(1+\frac{1}{2}+\cdots+\frac{1}{n-1}\right)\right)dt. \qquad (11) \end{aligned}$$

Then, by the right inequality in (8), we have

$$-x\left(1+\frac{1}{2}+\cdots+\frac{1}{n-1}\right) > -x(1+\log(n-1)) \quad \text{for all} \quad x\in(0,1]. \quad (12)$$

Inserting estimate (12) into (11), we get

$$n!|b_n| > (n-1)!\int_0^{r^{-1}} r^2 t\exp(-t(1+\log(n-1)))dt.$$

Integration by parts above then yields

$$\begin{aligned} n|b_n| \quad &> \quad \left(\frac{-re^{-(1+\log(n-1))}}{1+\log(n-1)} - \frac{r^2 e^{-(1+\log(n-1))}}{(1+\log(n-1))^2} + \frac{r^2}{(1+\log(n-1))^2}\right) \\ &= \quad \frac{r^2}{(1+L)^2}\left(1-\frac{e^{-1}}{n-1}\left(1+\frac{1+L}{r}\right)\right), \end{aligned}$$

where $L := \log(n-1)$.

Since it is easily established that $(1+L)/(n-1) \leqslant 1$ for all n (in fact, this follows from the left inequality (7) with $t := -\log(n-1)$), we have

$$n|b_n| > \frac{r^2}{(1+L)^2}\left(1-\frac{e^{-1}}{n-1}-\frac{e^{-1}}{1-e^{-1}}\right).$$

Therefore, when $n > 4$, we have

$$n|b_n| > \frac{1}{8(1+L)^2},$$

giving the left inequality of (6) and completing the proof. □

3 The log-concavity of $(s-1)\zeta(s)$

To prove our Theorem 1, we need to verify that $(s-1)\zeta(s)$ is log-concave for sufficiently large real numbers $s > 1$, where $\zeta(s)$ is the Riemann zeta function. We begin with the following elementary estimates.

Lemma 4. *We have $s\zeta(s) \leqslant s+1$ for $s \geqslant 3$, and $s^2\zeta(s) \leqslant s^2+1$ for $s \geqslant 5$.*

Proof. Using the series for $\zeta(s)$ as an approximation to an integral with rectangles from below, we get

$$\begin{aligned}\zeta(s) &= 1 + 2^{-s} + 3^{-s} + \cdots \\ &< 1 + 2^{-s} + \int_2^\infty x^{-s}\,dx \\ &= 1 + 2^{-s} + \frac{2^{1-s}}{s-1} \\ &\leqslant 1 + \frac{1}{s} \quad \text{for} \quad s \geqslant 3,\end{aligned}$$

giving the first statement. Since

$$1 + 2^{-s} + \frac{2^{1-s}}{s-1} \leqslant 1 + \frac{1}{s^2} \quad \text{for} \quad s \geqslant 5,$$

the second statement follows as well.

□

Let

$$f(s) := \log((s-1)\zeta(s)) \quad \text{for} \quad s > 1.$$

Lemma 5. *We have $f''(s) < 0$ for all $s \geqslant 6$.*

Proof. We first observe that

$$\begin{aligned}f'(s) &= \frac{d}{ds}\left(\log(s-1) + \log\zeta(s)\right) = \frac{1}{s-1} + \frac{\zeta'(s)}{\zeta(s)} \\ &= \frac{1}{s-1} - \sum_{n=2}^\infty \Lambda(n) n^{-s},\end{aligned}$$

where

$$\Lambda(n) = \begin{cases} \log p, & \text{if } n = p^m \text{ for some prime } p \text{ and integer } m \geqslant 1, \\ 0, & \text{otherwise}\end{cases}$$

is the von–Mangoldt function, and therefore

$$f''(s) = -\frac{1}{(s-1)^2} + \sum_{n=2}^\infty \Lambda(n)(\log n) n^{-s}$$

for all $s > 1$. Since $0 \leqslant \Lambda(n) \leqslant \log n < n^{1/2}$ for $n \geqslant 1$, we have by Lemma 4

$$\begin{aligned} f''(s) &= -\frac{1}{(s-1)^2} + \sum_{n=2}^{\infty} \Lambda(n)(\log n) n^{-s} \\ &< -\frac{1}{(s-1)^2} + \sum_{n=2}^{\infty} n \cdot n^{-s} \qquad (\text{for} \quad s > 2) \\ &= -\frac{1}{(s-1)^2} + (\zeta(s-1) - 1) \\ &\leqslant -\frac{1}{(s-1)^2} + \frac{1}{(s-1)^2} = 0 \quad \text{for} \quad s \geqslant 6, \end{aligned}$$

as desired. □

Remark. While not needed for our purposes, we can verify numerically that in fact $f''(s) < 0$ for all $s > 1$. To estimate $f''(s)$ for $s > 1$ and near 1, we use the formula ([2], p. 56)

$$\zeta(s) = \frac{s}{s-1} - s\int_1^\infty \frac{\{x\}}{x^{s+1}} dx := \frac{s}{s-1} - sg(s),$$

where $\{x\} = x - [x]$ denotes the fractional part of x and

$$g(s) := \int_1^\infty \frac{\{x\}}{x^{s+1}} dx.$$

Then

$$f(s) = \log((s-1)\zeta(s)) = \log\left(s - s(s-1)g(s)\right) = \log s + \log\left(1 - (s-1)g(s)\right).$$

For $1 < s < 2$, we have that

$$0 < (s-1)g(s) = (s-1)\int_1^\infty \frac{\{x\}}{x^{s+1}} dx < (s-1)\int_1^\infty \frac{dx}{x^2} = s - 1 < 1,$$

so we can expand the logarithm in series getting

$$f(s) = \log s - \sum_{k \geqslant 1} \frac{1}{k}(s-1)^k g(s)^k = \log s - (s-1)g(s) - \frac{1}{2}(s-1)^2 g(s)^2 + (s-1)^3 h(s),$$

where $h(s)$ is analytic in a neighborhood of 1. Taking derivatives we get

$$f'(s) = \frac{1}{s} - g(s) - (s-1)g'(s) - (s-1)g(s)^2 + (s-1)^2 a(s),$$

where $a(s) := -g(s)g'(s) + 3h(s) + (s-1)h'(s)$. So,

$$f''(s) = -\frac{1}{s^2} - 2g'(s) - g(s)^2 + (s-1)b(s),$$

where $b(s) := -g''(s) - 2g(s)g'(s) + 2a(s) + (s-1)a'(s)$. Thus, $f''(s)$ in a neighborhood of 1 is close to

$$-1 - 2g'(1) - g(1)^2 = -1 + 2\int_1^\infty \frac{\{x\}\log x}{x^2}dx - \left(\int_1^\infty \frac{\{x\}}{x^2}dx\right)^2,$$

a number which is about -0.19.

4 The log-concavity of $\alpha_n(x)$

In this section, we give the proof of Theorem 1. Log-concavity of $\alpha_n(x)$ is easily verified directly for $n \leqslant 4$, so in what follows we assume $n \geqslant 4$. The polynomial $\alpha_n(x) = a_0 + a_1 x + \cdots + a_m x^m$ has degree $m := [n/2]$, with coefficients $a_0 := n!|b_n|$ and $a_k := (n|B_{2k}|/2k)|s(n-1, 2k-1)|$ for $1 \leqslant k \leqslant m$. Log-concavity of $\alpha_n(x)$ is equivalent to the statement that the sequence of ratios of coefficients

$$\frac{a_0}{a_1}, \frac{a_1}{a_2}, \ldots, \frac{a_{m-1}}{a_m} \tag{13}$$

is nondecreasing.

For $\alpha_n(x)$ we have, by Lemma 3, that

$$\frac{a_0}{a_1} = \frac{n!|b_n|}{(n/2)|B_2 s(n-1,1)|} = 12(n-1)|b_n| < \frac{12}{(\log n)^2}. \tag{14}$$

By direct calculation, we have

$$\frac{a_1}{a_2} = \frac{4|B_2 s(n-1,1)|}{2|B_4 s(n-1,3)|} = 10\frac{|s(n-1,1)|}{|s(n-1,3)|}. \tag{15}$$

From equation (4.8) in [9], we have

$$\begin{aligned}|s(n-1,3)| &= |s(n-1,1)|\sum_{1\leqslant i<j\leqslant n-2}\frac{1}{ij} \\ &= \frac{|s(n-1,1)|}{2}\left(\left(\sum_{k=1}^{n-2}\frac{1}{k}\right)^2-\sum_{k=1}^{n-2}\frac{1}{k^2}\right) \\ &< \frac{|s(n-1,1)|}{2}((1+\log(n-2))^2-1), \qquad (16)\end{aligned}$$

where the last inequality above follows by using the left inequality of (8). Therefore, by estimates (14), (15), and (16), we have

$$\frac{a_0}{a_1}<\frac{12}{(\log n)^2}<\frac{20}{(1+\log(n-2))^2-1}<\frac{a_1}{a_2},$$

thus demonstrating the first inequality required in (13).

The log-concavity of $\alpha_n(x)$ will then be demonstrated by showing that the sequence

$$\left\{\frac{a_k}{a_{k+1}}\right\}=\left\{\frac{(2k+2)|B_{2k}s(n-1,2k-1)|}{2k|B_{2k+2}s(n-1,2k+1)|}\right\}$$

for $1\leqslant k<m$ is nondecreasing. From Theorem 3.1 in [8], we know that the sequence

$$\left\{\frac{1}{2k(2k-1)}\frac{|s(n-1,2k-1)|}{|s(n-1,2k+1)|}\right\}_{k\geqslant 1}$$

is an increasing sequence in k. It therefore suffices to show that the sequence

$$\left\{\frac{(2k+2)|B_{2k}|}{2k|B_{2k+2}|}\cdot 2k(2k-1)\right\}_{k\geqslant 1}$$

is also nondecreasing. From the well-known formula of Euler

$$\zeta(2k)=(-1)^{k+1}\frac{(2\pi)^{2k}B_{2k}}{2(2k)!} \qquad (17)$$

valid for all positive integers k, we have

$$\left\{\frac{(2k+2)|B_{2k}|}{2k|B_{2k+2}|}\cdot 2k(2k-1)\right\}=\left\{4\pi^2\frac{(2k-1)\zeta(2k)}{(2k+1)\zeta(2k+2)}\right\} \qquad (18)$$

for all $k \geqslant 1$. By means of equation (17) we verify directly that the sequence (18) is increasing for $1 \leqslant k \leqslant 3$. The fact that the sequence (18) is increasing for $k \geqslant 3$ follows from the fact that the function

$$g(s) := \frac{(s-1)\zeta(s)}{(s+1)\zeta(s+2)}$$

is increasing for $s \geqslant 6$, which is a direct consequence of the log-concavity of $(s-1)\zeta(s)$ for $s \geqslant 6$ demonstrated in Lemma 5. Therefore the sequence (18) is increasing for all $k \geqslant 1$, thus completing the proof of our Theorem 1.

5 Estimation of the roots of $\beta_n(\lambda)$

In this section, we prove Corollary 2. The given bounds are easily verified directly for $n < 4$, so in what follows we assume $n \geqslant 4$. For this, we invoke the theorem of Kakeya ([7],[4]), which says that for a polynomial $p(x) := a_0 + a_1 x + \cdots + a_m x^m$ with positive real coefficients a_i, the minimum value in

$$\frac{a_0}{a_1}, \frac{a_1}{a_2}, \ldots, \frac{a_{m-1}}{a_m}$$

is a lower bound, and the maximum such value is a upper bound, for the absolute value of a root of $p(x)$. For $p(x) = \alpha_n(x)$, we have shown that the minimum such value is

$$\frac{a_0}{a_1} = 12(n-1)|b_n|.$$

By Lemma 3, we have

$$12(n-1)|b_n| > \frac{12(n-1)}{8n(1+\log(n-1))^2} > \frac{1}{(1+\log(n-1))^2}. \tag{19}$$

Thus, every root x of $\alpha_n(x)$ satisfies $|x| \geqslant (1+\log(n-1))^{-2}$. Since a nonzero value λ is a root of $\beta_n(\lambda)$ if and only if $x = -1/\lambda^2$ is a root of $\alpha_n(x)$ (see transformation (4)), it follows that every root λ of $\beta_n(\lambda)$ satisfies $|\lambda| \leqslant 1 + \log(n-1)$, giving our stated upper bound.

By Theorem 1, if $n = 2m$ is even, the largest of our coefficient ratios is

$$\frac{a_{m-1}}{a_m} = \frac{2m|B_{2m-2}s(2m-1, 2m-3)|}{(2m-2)|B_{2m}s(2m-1, 2m-1)|}. \tag{20}$$

It is immediate from estimate (2) that $s(n-1,n-1)=1$, and

$$\begin{aligned} s(n-1,n-3) &= \sum_{1\leqslant i<j\leqslant n-2} ij \\ &= \frac{1}{2}\left(\left(\sum_{k=1}^{n-2} k\right)^2 - \sum_{k=1}^{n-2} k^2\right) \\ &= \frac{1}{2}\left(\left(\frac{(n-1)(n-2)}{2}\right)^2 - \frac{(n-1)(n-2)(2n-3)}{6}\right) \\ &< \frac{(n-1)^2(n-2)^2}{8}. \end{aligned} \tag{21}$$

From Euler's formula (17), we have

$$\frac{2m|B_{2m-2}|}{(2m-2)|B_{2m}|} = \frac{4\pi^2\zeta(2m-2)}{(2m-1)(2m-2)\zeta(2m)}, \tag{22}$$

so putting together (20), (21) (recall that $n=2m$), and (22), we arrive at

$$\frac{a_{m-1}}{a_m} < \frac{\pi^2}{2}(2m-1)(2m-2)\frac{\zeta(2m-2)}{\zeta(2m)}. \tag{23}$$

Since $\zeta(2m)>1$ and $(2m-2)\zeta(2m-2)\leqslant(2m-1)$ for $m>2$ by Lemma 4, we get from inequality (23), that

$$\frac{a_{m-1}}{a_m} < \frac{\pi^2(2m-1)^2}{2}.$$

Thus, by Kakeya's theorem mentioned above, every root x of $\alpha_n(x)$ satisfies $|x|\leqslant\pi^2(n-1)^2/2$, which implies, via transformation (4), that every root λ of $\beta_n(\lambda)$ satisfies $|\lambda|\geqslant\sqrt{2}/(\pi(n-1))$, when n is even.

By Theorem 1, if $n=2m+1$ is odd, the largest of our coefficient ratios is

$$\frac{a_{m-1}}{a_m} = \frac{2m|B_{2m-2}s(2m,2m-3)|}{(2m-2)|B_{2m}s(2m,2m-1)|}. \tag{24}$$

It is immediate from (2) that $|s(n-1,n-2)| = (n-1)(n-2)/2$, and

$$\begin{aligned} |s(n-1,n-4)| &= \sum_{1\leqslant i<j<k\leqslant n-2} ijk \\ &< \frac{1}{6}\left(\sum_{k=1}^{n-2} k\right)^3 \\ &= \frac{(n-1)^3(n-2)^3}{48}, \end{aligned}$$

and therefore

$$\frac{|s(n-1,n-4)|}{|s(n-1,n-2)|} < \frac{(n-1)^2(n-2)^2}{24}. \tag{25}$$

Now combining (24), (25) (recall that $n = 2m+1$), and (22), we get

$$\frac{a_{m-1}}{a_m} < \frac{\pi^2}{6}(2m-1)(2m-2)\frac{\zeta(2m-2)}{\zeta(2m)}. \tag{26}$$

Since $\zeta(2m) > 1$ and $(2m-2)\zeta(2m-2) \leqslant (2m-1)$ for $m > 2$ again by Lemma 4, we get from estimate (26) that

$$\frac{a_{m-1}}{a_m} < \frac{\pi^2(2m-1)^2}{6}.$$

Thus, every root x of $\alpha_n(x)$ satisfies $|x| \leqslant \pi^2(n-2)^2/6$, which implies, again via transformation (4), that every nonzero root λ of $\beta_n(\lambda)$ satisfies $|\lambda| \geqslant \sqrt{6}/(\pi(n-2))$, when n is odd. This completes the proof of Corollary 2.

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