# Regular and Strongly Regular Planar Graphs

*Nirmala B. Limaye*[1], *Dinesh G. Sarvate*[2], *Pantelimon Stănică*[3*], *Paul T. Young*[2]

[1] Dept. of Mathematics, University of Mumbai, Vidyanagari, Mumbai 400 098, India
[2] Dept. of Mathematics, College of Charleston, Charleston, SC 29424, USA
[3] Dept. of Mathematics, Auburn University Montgomery, Montgomery, AL 36124, USA

October 26, 2004

We give a constructive proof that a planar graph on $n$ vertices with degree of regularity $k$ exists for all pairs $(n, k)$ except for two pairs $(7, 4)$ and $(14, 5)$. We continue this theme by classifying all strongly regular planar graphs, and then consider a new class of graphs called 2-*strongly regular*. We conclude with a conjectural classification of all planar 2-strongly regular graphs.

## 1 Introduction

In the first two parts of this paper we investigate regular and strongly regular graphs that are planar, classifying such graphs completely. Finally, we define a new type of graph, which we call 2-strongly regular graph (or, in general $l$-strongly regular graph) which satisfies the same requirement as a strongly regular graph, allowing, however, two possible degrees (or, in general $l$ degrees).

We shall assume that all of our graphs are connected. A graph is a $k$-*regular graph* if every vertex has degree $k$. Let $\delta(x, y)$ be the number of vertices adjacent to both $x, y$. We say that a $k$-regular graph $G$ on $n$ vertices is a *strongly regular graph* (SRG) with parameter set $(n, k, \lambda, \mu)$, denoted by $srg(n, k, \lambda, \mu)$, if there exist nonnegative integers $\lambda, \mu$ such that for all vertices $u$, $v$ the number of vertices adjacent to both $u, v$ is $\lambda$ (respectively, $\mu$), if $u, v$ are adjacent (respectively, nonadjacent). For more definitions, the reader might want to consult [1].

## 2 Regular Planar Graphs

It has been well known that for positive integers $n$ and $k < n$, there exists an $k$-regular graph on $n$ vertices if and only if $nk$ is even. So it is natural to ask whether under those conditions a regular planar graph on $n$ vertices exists. Since the minimum degree $\delta(G)$ of a planar graph is at most 5, one must have $0 \leq k \leq 5$. For these basic results one can refer to any text book on graph theory, for example [2].

For $k = 0, 1, 2$ isolated points, parallel edges and cycles answer this question affirmatively.
*Case $k = 3$:* For $n = 4$, we have $K_4$. We can then construct bigger graphs inductively. First of all, $n$ has to be even. If $G_{2n}$ is a 3-regular graph on $2n$ vertices, consider its outer boundary. Let $x, w, y$ be a path of length 2 on this outer boundary. Take a new vertex $x_1$ on the edge $xw$ and a new vertex $y_1$ on the edge $wy$. Add an edge $x_1y_1$ using a curve completely in the outer region of

*Also associated with the Institute of Mathematics "Simion Stoilow" of the Romanian Academy, Bucharest - Romania.

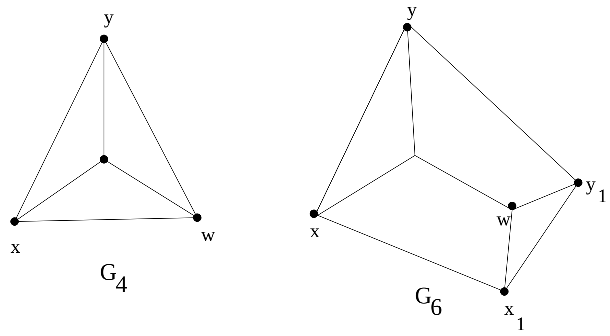



Figure 1: (4,3) and (6,3) graphs

$G_{2n}$. One can easily see that this is a 3-regular planar graph on $2n+2$ vertices. This procedure is illustrated in Figure 1.
*Case $k=4$:* If $n=2t, t \geq 6$, we can take a cycle $C=\{v_1, v_2, \cdots, v_{2t}, v_1\}$. Then we add two cycles $\{v_1, v_3, \cdots, v_{2t-1}, v_1\}$ and $\{v_2, v_4, \cdots, v_{2t}, v_2\}$, one in the interior region and one in the exterior region of $C$.

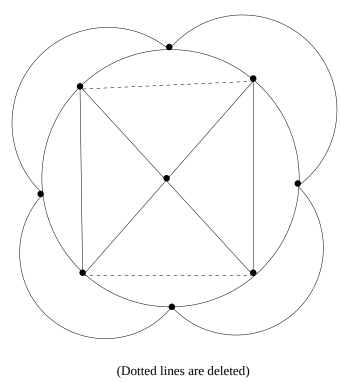



Figure 2: (9,4) graph

If $n=2t+1$, we must have $t>2$ since $K_5$ is not planar. First, we take $n \geq 9$. We can then take the 4-regular graph on $2t$ vertices described before. Inner cycle has at least 4 vertices. That means we can select two parallel edges in this inner cycle. We remove them and take an extra vertex in the interior region of this cycle and join it to the end vertices of the edges which are removed. This will give a 4-regular planar graph on $2t+1$ vertices. This is illustrated in Figure 2.

**Proposition 1.** *If $G$ is a 4-regular graph on 7 vertices, then it cannot be planar.*

*Proof.* If is enough to show that $G$ contains a homeomorph of $K_{n,m}$ for some $n \geq 3, m \geq 3$.

Clearly $G$ is not $K_7$, therefore it has a pair $v_1, v_2$ of nonadjacent vertices. Let the other vertices be $v_3, \cdots, v_7$. Since $v_1 v_2 \notin E(G)$, $v_1$ and $v_2$ must have at least 3 common neighbours.
*Case 1:* $v_1, v_2$ are both adjacent to $v_3, v_4, v_5, v_6$. This means that $v_7$ is also adjacent to $v_3, v_4, v_5, v_6$ and we have a copy of $K_{3,4}$ in $G$.
*Case 2:* $v_1, v_2$ have only three common neighbours. Let those be $v_3, v_4, v_5$. Without loss of generality, we can assume that $v_1 v_6, v_2 v_7 \in E(G)$.

If $v_6, v_7$ are nonadjacent, they must be adjacent to $v_3, v_4, v_6$ giving a copy of $K_{3,4}$ on the partition $\{v_3, v_4, v_5\} \bigcup \{v_1, v_2, v_6, v_7\}$.

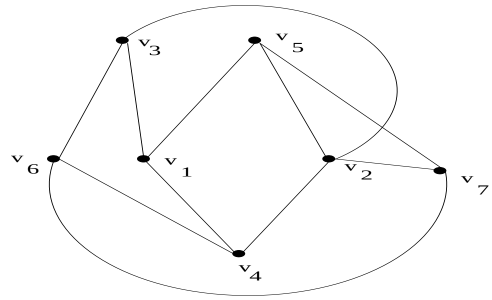



Figure 3: Homeomorph of $K_{3,3}$

If $v_6 v_7 \in E(G)$, then $v_6$ is adjacent to two of $v_3, v_4, v_5$. But then $v_5 v_7 \in E(G)$ as shown in the figure 3. This is clearly a homeomorph of $K_{3,3}$. Thus $G$ is not planar. $\square$

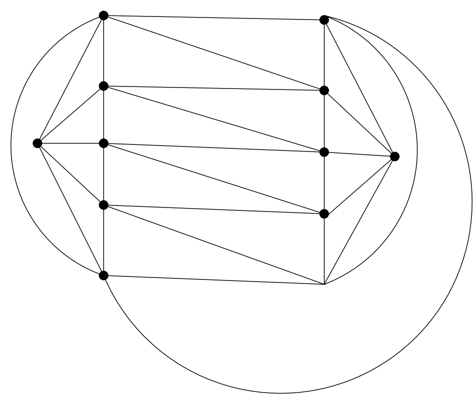


Figure 4: (12,5) graph

*Case $k = 5$:* If we want a 5-regular planar graph on $n$ vertices, then $n$ must be even. Moreover $\frac{5n}{2} \leq 3n - 6$. This gives $n \geq 12$. Figure 4 gives a 5-regular graph on twelve vertices.

Before we proceed, we need following lemma:

**Lemma 2.** *If there exists a 5-regular planar graph on $n$ vertices, then there exists a 5-regular planar graph on $n + 10$ vertices.*

*Proof.* Let $G$ be a 5-regular planar graph on $n$ vertices. Let $x$ be a vertex of $G$ with neighbours $x_1, x_2, x_3, x_4, x_5$. Remove the vertex $x$ and replace it by a configuration as in Figure 5.

The new vertices are $y_1, y_2, \cdots, y_5, z_1, z_2, \cdots, z_5, \tilde{x}$ and the new edges are $x_i\ y_i$, $z_i\ y_i$, $z_i\ y_{i-1}$, $y_{i-1}\ y_i$, $z_{i-1}\ z_i$, $1 \leq i \leq 5$; here $i - 1$ is taken modulo 5, and $\tilde{x}\ z_i, 1 \leq i \leq 5$.

One can see that this produces a 5-regular planar graph on $n + 10$ vertices. $\square$

This means that we need to find the required planar graphs for $n = 12, 14, 16, 18, 20$. The value $n = 12$ is already cleared.

**Proposition 3.** *There is no planar, 5-regular graph on* 14 *vertices.*

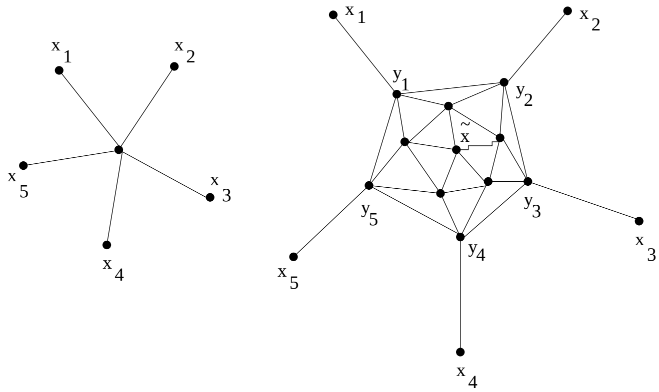



Figure 5: Configuration for Lemma 2

*Proof.* Suppose $G$ is a planar graph on 14 points which is 5-regular. This has 35 edges. This means that exactly one region is a quadrangle and all the other regions are triangles. In fact we can draw the graph in such a way that the outer region is a quadrangle with the boundary $\{x, y, z, w, x\}$ (say). Now each of the edges $xy, yz, zw, wx$ must belong to one more triangular region.

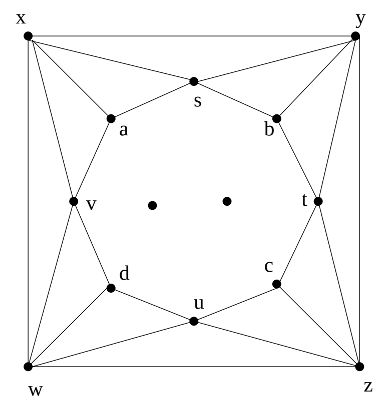



Figure 6: (14,5) configuration

Let these regions be $xys, yzt, zwu, wxv$. One can easily chech that the vertices $s, t, u, v$ must be all distinct. Now each of the vertices $x, y, z, w$ have one more neighbour. Let those be $a, b, c, d$ respectively. Again these have to be distinct for otherwise one cannot complete the required degrees of some of the earlier vertices and still keep the graph planar. Since all the remaining regions are triangles one can see that graph $G$ must have the configuration shown in Figure 6 and then it is impossible to complete the construction to make the degrees of the enclosed two vertices equal to 5 and still keep the graph planar. Hence a planar graph for the pair $(14, 5)$ does not exist. □

This means the value 24 has to be considered separately along with 16, 18 and 20. Figure 7 gives a 5-regular planar graph on 16 and 20 points.

The value $n = 18$ is cleared by Figure 8.

Finally, for $n = 24$, take two copies $G$ and $G'$ of 5-regular graph on twelve vertices. Let $xy$ and $x'y'$ be two corresponding edges on the outer boundary. Remove these edges and add edges $x\,x', y\,y'$. The resulting graph is 5-regular on 24 points. Thus, for each even number $n \geq 12, n \neq 14$, there exists a 5-regular graph of order $n$.

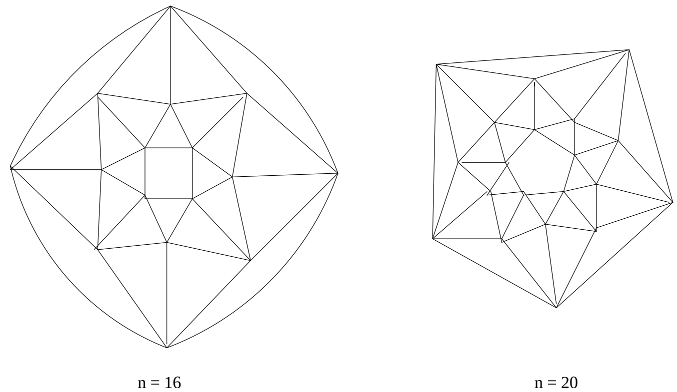



Figure 7: (16,5) and (20,5) Graphs

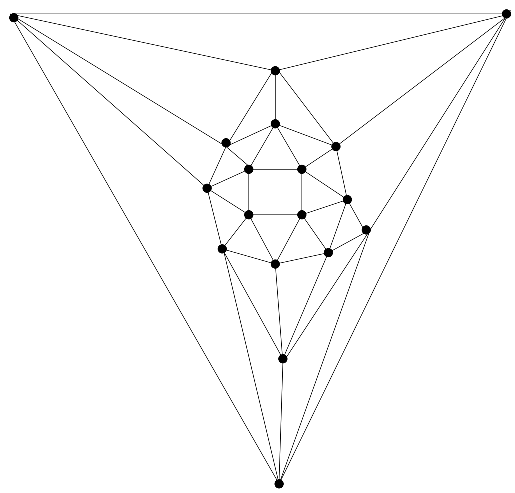


Figure 8: (18,5) Graph

# 3 Strongly Regular Planar Graphs

Having proved that all planar connected regular graphs of degree $k \leq 5$ exists on $n$ vertices except when $n = 7$, $k = 4$ and $n = 14$ and $k = 5$, the next natural question is to determine which connected strongly regular graphs are planar. Let $\Gamma(x)$ and $\Delta(x)$ be the sets of vertices adjacent to $x$, respectively, nonadjacent to an arbitrary vertex $x$. Counting in two ways the number of edges between $\Gamma(x)$ and $\Delta(x)$ yields the following very useful (and well-known) lemma.

**Lemma 4.** *The parameters $(n, k, \lambda, \mu)$ of a SRG satisfy*

$$k(k - \lambda - 1) = (n - k - 1)\mu. \tag{1}$$

Our main result on this question is the following theorem.

**Theorem 5.** *Except for the octahedral circulant graph $srg(6, 4, 2, 4)$, the cycle graphs $C_n$, $n = 4, 5$, and the complete graphs $K_1, K_2, K_3, K_4$ on $1, 2, 3$ and $4$ vertices, all other SRGs are nonplanar.*

*Proof.* By abuse of notation, we shall use $srg(\cdot, \cdot, \cdot, \cdot)$ also for a possible parameter set of a SRG. Employing Lemma 4, we show that there are only a finite number of possible planar strongly regular graphs, namely we shall prove that the following are the only (connected) planar strongly regular graphs

$$\begin{aligned}
&srg(1,0,0,0) = K_1 \text{ (singleton)};\ srg(2,1,0,0) = K_2 \text{ (path)};\\
&srg(3,2,1,0) = K_3 \text{ (complete)};\ srg(4,2,0,2) = C_4 \text{ (cycle)};\\
&srg(4,3,2,0) = K_4 \text{ (complete)};\ srg(5,2,0,1) = C_5 \text{ (cycle)};\\
&srg(6,4,2,4) = Ci_6(1,2) \text{ (octahedral circulant)};
\end{aligned} \tag{2}$$

Assume that $G$ is a SRG, with parameter sets $(n, k, \lambda, \mu)$. One knows that if a graph $G$ is planar then its degrees are less than or equal to 5, so $k \leq 5$. Now, we will apply Lemma 4 for each of the six possible values of $k$:

*Case* $k = 0$. We obtain the totally disconnected graph $srg(n, 0, 0, 0)$, unless $n = 1$.

*Case* $k = 1$. Thus, $\lambda = (2 - n)\mu$. Therefore, the only possible parameter sets for a SRG are $srg(1, 1, \lambda, \lambda)$, an impossibility, or $srg(2, 1, 0, 0) = K_2$.

*Case* $k = 2$. Thus, we need $2(1 - \lambda) = (n - 3)\mu$. This constraint on the parameters renders the possibilities: $srg(5, 2, 0, 1) = C_5$ & $srg(4, 2, 0, 2) = C_4$ *(cycles)*, $srg(3, 2, 1, \mu) = K_3$, $srg(n, 2, 1, 0)$ $(n > 3)$ (the last turns out to be a disjoint union of triangles and hence not connected).

*Case* $k = 3$. Thus, we need $3(2 - \lambda) = (n - 4)\mu$. By arithmetical reasoning we derive the possibilities: $srg(4, 3, 2, \mu)$ for $\mu \leq 2$, $srg(5, 3, 1, 3)$, $srg(7, 3, 1, 1)$, $srg(5, 3, 0, 6)$, $srg(6, 3, 0, 3)$, $srg(7, 3, 0, 2)$, $srg(10, 3, 0, 1)$.

*Case* $k = 4$. Thus, we need $4(3 - \lambda) = (n - 5)\mu$. By congruence considerations, we arrive at the possibilities: $srg(n, 4, 3, 0)$, $srg(5, 4, 3, \mu)$ $(\mu \leq 3)$, $srg(6, 4, 0, 12)$, $srg(6, 4, 1, 8)$, $srg(6, 4, 2, 4)$, $srg(7, 4, 0, 6)$, $srg(7, 4, 1, 4)$, $srg(7, 4, 2, 2)$, $srg(8, 4, 0, 4)$, $srg(9, 4, 0, 3)$, $srg(9, 4, 1, 2)$, $srg(9, 4, 2, 1)$, $srg(11, 4, 0, 2)$, $srg(13, 4, 1, 1)$, $srg(17, 4, 0, 1)$.

*Case* $k = 5$. Thus, we need $5(4 - \lambda) = (n - 6)\mu$. As before, we have the possibilities: $srg(n, 5, 4, 0)$, $srg(6, 5, 4, \mu)$ $(\mu \leq 4)$, $srg(7, 5, 0, 20)$, $srg(7, 5, 1, 15)$, $srg(7, 5, 2, 10)$, $srg(7, 5, 3, 5)$, $srg(8, 5, 0, 10)$, $srg(8, 5, 2, 5)$, $srg(9, 5, 1, 5)$, $srg(10, 5, 0, 5)$, $srg(11, 5, 0, 4)$, $srg(11, 5, 2, 2)$, $srg(11, 5, 1, 3)$, $srg(11, 5, 3, 1)$, $srg(16, 5, 0, 2)$, $srg(16, 5, 2, 1)$, $srg(21, 5, 1, 1)$, $srg(26, 5, 0, 1)$.

Now, by Kuratowski's theorem, a graph $G$ is planar if and only if it has no subgraphs isomorphic to subdivisions of the complete graph $K_5$ or the bipartite graph $K_{3,3}$. If $n > 6$, the graphs having parameters sets

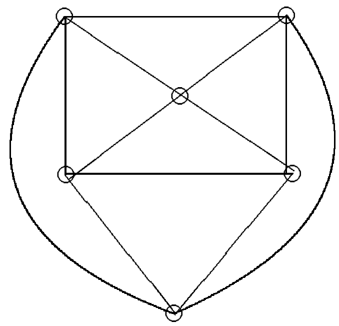
$$srg(n, 2, 1, 0);\ srg(n, 4, 3, 0);\ srg(n, 5, 4, 0) \tag{3}$$

cannot be (connected) strongly regular as the following analysis shows. If $srg(n, k, k - 1, 0)$ exists, it must contain a complete graph on $k + 1$ vertices: Let $x$ be a vertex with $k$ adjacent vertices. As $\mu = 0$, they must be adjacent to each other. In case of $k = 2$, we have a planar connected graph only when $n = 3$, but for $k \geq 4$, we have a component containing $K_5$ and hence is nonplanar. Therefore, the only *possible* connected planar SRGs (among the ones listed in the five cases above) are

$$\begin{aligned}
&srg(1,0,0,0) \text{ (singleton)};\ srg(2,1,0,0) = P_2 \text{ (path)};\\
&srg(3,2,1,0) = K_3 \text{ (complete)};\ srg(4,2,0,2) = C_4 \text{ (square cycle)};\\
&srg(4,3,2,0) = K_4 \text{ (complete)};\ srg(5,2,0,1) = C_5 \text{ (pentagon cycle)};\\
&srg(5,4,3,0) = K_5 \text{ (complete)};\ srg(6,3,0,3) = Ci_6(1,3) \text{ (circulant)};\\
&srg(6,4,2,4) = Ci_6(1,2) \text{ (octahedral)};\ srg(6,5,4,0) = K_6 \text{ (complete)};\\
&srg(8,4,0,4) = Ci_8(1,3) \text{ (circulant)};\ srg(10,3,0,1) = P \text{ (Petersen)};\\
&srg(16,5,0,2) \text{ (Clebsch)};
\end{aligned} \tag{4}$$

Certainly, the first six graphs are planar as one can see easily.

By the same theorem of Kuratowski, $K_5, K_6$ are nonplanar. The famous Petersen and Clebsch graphs are certainly nonplanar (see [3, 5]). The octahedral graph is planar as one can see next



Certainly, $Ci_6(1,3)$ and $Ci_8(1,3)$, which are unique and contain a $K_{3,3,}$ are nonplanar: assume that vertex $v_1$ is adjacent to $v_2, v_3, v_4$. Now $v_2$ has to be adjacent to $v_1, v_5, v_6$, and $v_3$ has to be adjacent to $v_1, v_5, v_6$. Similarly for $v_4$. We have the theorem. $\square$

**Remark 6.** *A similar argument as the last one reveals that any $srg(2n, n, 0, n)$ is unique and contains a $K_{3,3,}$, $n \geq 3$ therefore it is nonplanar.*

## 4 2-strongly regular graphs

**Definition 7.** *A connected graph is called 2-strongly regular (2-SRG) with parameters $(n, r_1 < r_2, \lambda, \mu)$ if every vertex has two possible degrees $r_1 < r_2$, and $\delta(x, y) = \lambda$, respectively, $\mu$ if $x, y$ are adjacent, respectively, nonadjacent.*

In other words, we impose a strongly regular-like condition without the regularity. One might suspect that there should be more 2-SRGs than SRGs, since we allow two possible degrees to occur. Later we will conjecture that surprisingly, there is essentially one construction (yielding an infinite class on nonisomorphic 2-SRGs, though).

We start with some observations gathered in a proposition.

**Proposition 8.** *Let $G$ be a 2-SRG. Then, between any two vertices there is a path of length at most 2. Moreover, a cycle $C_n$, $n \geq 4$, cannot be a component in a point-union of a 2-SRG.*

*Proof.* Let $v, w$ be two arbitrary vertices of $G$. If $v, w$ are adjacent, there is nothing to prove. Assume now that there are two vertices $v, w$ with a minimal path of length at least three (so $v, w$ are nonadjacent), given by $v, c_1, c_2, \ldots, w$. Obviously, $\delta(v, c_2) \geq 1$ (since $c_1$ is a common vertex to $v$ and $c_2$). That implies that $\mu \geq 1$, so $\delta(v, w) = \mu \geq 1$. Thus, there is a common vertex to $v$ and $w$. That contradicts the minimality of the path $v, c_1, c_2, \ldots, w$.

To prove the second claim, let $n \geq 4$. We assume that there is a cycle $C_n = < a_1, a_2, \ldots, a_n >$, $n \geq 4$ with $a_1$ a contact point in $G$. Let $x \in G - C_n$ adjacent to $a_1$. It follows that $\delta(x, a_2) = \mu \geq 1$. But $\delta(x, a_3) = 0$, which is a contradiction. $\square$

Our next result shows a diophantine relation among the parameters of a 2-SRG similar to the one of a SRG.

**Theorem 9.** *Let $G$ be a 2-SRG of parameters $n, \lambda, \mu, r_1 < r_2$. Pick a vertex $x$ of degree $r_2$. If exactly $\alpha \geq 0$ of its neighbors have degree $r_2$ then*

$$(n - r_2 - 1)\mu = r_2(r_1 - \lambda - 1) + \alpha(r_2 - r_1). \quad (5)$$

*Pick a vertex $y$ of degree $r_1$. If exactly $\beta \geq 0$ of its neighbors have degree $r_2$ then*

$$(n - r_1 - 1)\mu = r_1(r_1 - \lambda - 1) + \beta(r_2 - r_1). \tag{6}$$

*Moreover, $\alpha$ and $\beta$ are independent of the considered vertices, and*

$$\beta = r_1 + \alpha + \mu - \lambda - 1. \tag{7}$$

*Proof.* We prove the theorem using a similar idea as for the classical SRGs. We take any vertex $x$ of maximal degree $r_2$. Let $\Gamma(x)$ and $\Delta(x)$ be the set of vertices adjacent, respectively, nonadjacent to $x$. We count the edges between $\Gamma(x)$ and $\Delta(x)$. Certainly, $\Delta(x)$ contains exactly $n - r_2 - 1$ vertices. For each of these vertices there are exactly $\mu$ common vertices between them and $x$, which vertices must be in $\Gamma(x)$. We obtain $(n - r_2 - 1)\mu$ edges between $\Gamma(x)$ and $\Delta(x)$.

Let $v$ be one of the $\alpha$ vertices of degree $r_2$ adjacent to $x$. Since $\delta(x, v) = \lambda$, it follows that exactly $\lambda$ neighbors for $v$ (also common to $x$) that are in $\Gamma(x)$, and the rest of $(r_2 - \lambda - 1)$ must be in $\Delta(x)$ (thus $\alpha(r_2 - \lambda - 1)$ edges between $\Delta(x)$ and $\Gamma(x)$). Similarly, for each of the $r_2 - \alpha$ vertices of degree $r_1$, producing $(r_2 - \alpha)(r_1 - \lambda - 1)$ more edges between $\Delta(x)$ and $\Gamma(x)$. This analysis renders the equation

$$(n - r_2 - 1)\mu = (r_2 - \alpha)(r_1 - \lambda - 1) + \alpha(r_2 - \lambda - 1),$$

which by simplification produces the first claim. The proof of the second equation is similar.

Regarding the further claim of our theorem, the equations (5) and (6) are linear equations in $\alpha$, respectively, $\beta$. Since their leading coefficient is $(r_2 - r_1) \neq 0$, each equation has a unique solution.

Solving the system given by both (5) and (6) we obtain

$$\mu = \frac{r_2\beta - r_1\alpha}{n - 1} \tag{8}$$

$$\lambda = -\frac{n - 1 + r_1 - nr_1 + \alpha - n\alpha + r_1\alpha - r_2\beta + n\beta - \beta}{n - 1}. \tag{9}$$

Simplifying the expression of $\lambda$, we obtain (7). □

A vertex $w$ whose neighbors are all of degree $r$ is called an *$r$-island.* In some cases, one can find a stronger relation among the parameters of a SRG.

**Theorem 10.** *Let $G$ be a 2-SRG of parameters $(n > 1, r_1 < r_2, \lambda, \mu)$. If there is a vertex $x$ of degree $r_2$ that is an $r_1$-island, then*

$$(n - r_2 - 1)\mu = r_2(r_1 - \lambda - 1); \tag{10}$$

*a vertex $y$ of degree $r_1$ that is an $r_2$-island, then*

$$(n - r_1 - 1)\mu = r_1(r_2 - \lambda - 1). \tag{11}$$

*Moreover, there is no vertex of degree $r_1$ that is an $r_1$-island, or a vertex of degree $r_2$ that is an $r_2$-island. Furthermore, the two cases* (10), (11) *are mutually exclusive.*

*Proof.* The equations are obtained by replacing $\alpha = 0$ in (5) and $\beta = r_1$ in (6).

Now assume that we have both a vertex $x$ of degree $r_2$ that is an $r_1$-island, and a vertex $y$ of degree $r_1$ that is an $r_2$-island. It follows that

$$(n - r_2 - 1)\mu = r_2(r_1 - \lambda - 1),$$
$$(n - r_1 - 1)\mu = r_1(r_2 - \lambda - 1).$$

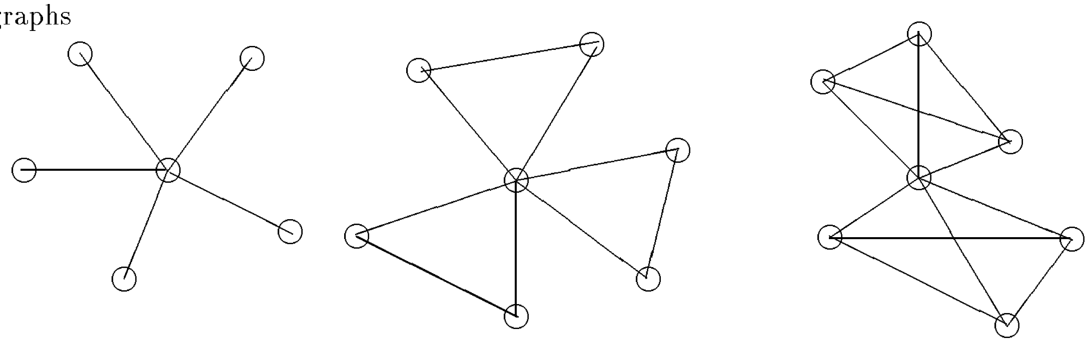
Solving the previous system for $\lambda, \mu$, we obtain

$$\lambda = r_1 + r_2 - n, \ \mu = r_2.$$

It follows that every nonadjacent pair of vertices have exactly $r_2$ common neighbors. That is impossible since there is at least a vertex $z$ of degree $r_1$ that cannot be adjacent to $y$.

There cannot be any vertex of degree $r_1$ that is an $r_1$-island, since that will force $\beta = 0$, which implies that $\mu < 0$, unless $\alpha = 0$. If $\alpha = 0$, then there is a (in fact any) vertex of degree $r_2$ that is an $r_1$-island. But then, that will force any of its neighbors to have $\beta \neq 0$, which is a contradiction. Certainly a vertex of degree $r_2$ cannot be an $r_2$-island since then $\alpha$ would be equal to $r_2$. But, then every vertex of degree $r_2$ would have only neighbors of degree $r_2$ and the graph would be disconnected (since it contains vertices of degree $r_1$, as well). □

The previous theorem suggests some examples of 2-SRGs, for instance, one-point union of complete graphs



We use the notation $K_n^{(s)}$ for the one-point union of $s$ copies of $K_n$. Thus, the previous graphs are $K_2^{(5)}, K_3^{(3)}, K_4^{(2)}$.

Based on our observations and extensive computations, we make the following

**Conjecture 11.** *Any 2-SRG is a one-point union of complete graphs.*

There are more conditions that the parameters of a 2-SRG must satisfy. We will deduce some in our next result.

**Theorem 12.** *Let $G$ be a 2-SRG. Then the number of vertices of degree $r_1$, respectively, $r_2$ is*

$$\frac{n(r_2 - \alpha)}{\beta - \alpha + r_2}, \ \textit{respectively}, \ \frac{n\beta}{\beta - \alpha + r_2}.$$

*Proof.* Take $A_1$, respectively, $A_2$ to be the sets of vertices of degree $r_1$, respectively, $r_2$. We will count the number of edges between $A_1$ and $A_2$ in two ways (assuming Theorem 9). Let $s$ be the number of elements of $A_1$. For each vertex, say $a_1$ in $A_1$, there are exactly $\beta$ vertices in $A_2$ adjacent to $a_1$. Thus, we obtain $s\beta$ edges between $A_1$ and $A_2$. Similarly, for each vertex, say $a_2$ in $A_2$, there are precisely $r_2 - \alpha$ vertices in $A_1$ adjacent to $a_2$. Therefore, we obtain $(n - s)(r_2 - \alpha)$ edges between $A_1$ and $A_2$. It follows that $s\beta = (n - s)(r_2 - \alpha)$, from which we deduce $s = \dfrac{n(r_2 - \alpha)}{\beta - \alpha + r_2}$. The second claim is implied by the fact that the number of vertices is $n$. □

It is well-known that for (connected and not complete) SRGs of parameters $(n, k, \lambda, \mu)$, we have the following inequalities

$$1 \leq \lambda + 1 < k, \ 0 < \mu < k < n - 1.$$

We prove some analogous inequalities for a 2-SRG graph.

**Proposition 13.** *Let $G$ be a 2-SRG. Then*

$$1 \leq \lambda + 1 \leq r_1, \ 0 < \mu < r_2 \leq n - 1.$$

*Proof.* Certainly $\lambda \geq 0$. Now, take a vertex $x$ of degree $r_2$ and one of its neighbors $y$ of degree $r_1$ (since $x$ is not an $r_2$-island). Then, the $\lambda$ common neighbors of $x$ and $y$ have to be among the $r_1 - 1$ vertices other than $x$ which are adjacent to $y$. Therefore, $\lambda + 1 \leq r_1$.

Now we prove the second inequality. If $\mu = 0$, Proposition 8 implies that every two vertices are adjacent, so we are dealing with a complete regular graph, which cannot happen for a 2-SRG. If $\mu = r_2$, then there are two vertices having exactly $r_2$ neighbors. But then, the two vertices would have at least degree $r_2 + 1$, which is impossible. □

**Remark 14.** *For classical strongly regular graphs, $k$ cannot be $n - 1$. However, for a 2-SRG, $r_2$ can be $n - 1$, for instance in the case of one-point union of $K_n$'s.*

Our next result gathers some inequalities and other divisibility conditions on the parameters, some of which generalize the previous ones.

**Theorem 15.** *For a 2-SRG $G$, the following divisibilities are true*

$$r_2 + r_1 + \mu - \lambda - 1 \mid n(r_2 - \alpha), \tag{12}$$
$$n - 1 \mid r_2\beta - r_1\alpha. \tag{13}$$

*Moreover, the following inequalities hold*

$$r_2\beta \geq r_1\alpha + n - 1, \tag{14}$$
$$r_2 \geq \alpha \geq \frac{n - 1 + (\lambda + 1 - r_1 - \mu)r_2}{r_2 - r_1}, \tag{15}$$
$$r_1 \geq \beta \geq \frac{n - 1 + (\lambda + 1 - r_1 - \mu)r_1}{r_2 - r_1}. \tag{16}$$

*Proof.* The divisibility claim follows immediately from the expression (8) of $\mu$ and Theorem 12. The inequality (14) follows from the divisibility $n - 1 \mid r_2\beta - r_1\alpha$, by observing that $r_2\beta \neq r_1\alpha$ (it would imply that $\mu = 0$, which cannot happen under our conditions).

Certainly $\alpha \leq r_2$ and $\beta \leq r_1$. Using Theorem 9 and inequality (14), we obtain

$$r_2(r_1 + \alpha + \mu - \lambda - 1) \geq r_1\alpha + n - 1,$$

therefore $r_2 \geq \alpha \geq \dfrac{n - 1 + (\lambda + 1 - r_1 - \mu)r_2}{r_2 - r_1}$.

For the second inequality, one uses $\beta \geq \dfrac{r_1\alpha + n - 1}{r_2}$ and the obtained inequality for $\alpha$. □

**Remark 16.** *In fact, $\alpha < r_2$, otherwise we would have a vertex of degree $r_2$ that is an $r_2$-island.*

Regarding planarity for 2-SRGs, we have the following result (the proof is straightforward).

**Theorem 17.** *Assuming Conjecture 11 true, the only planar 2-SRGs are one-point unions of complete graphs $K_2, K_3, K_4$, namely*

$$K_2^{(s)}; \ K_3^{(s)}; \ K_4^{(s)}.$$

One can extend the new type of graph, which we called 2-SRG, to a strongly regular-like graph with parameters $n, \lambda, \mu$, where we force exactly $l$ degrees on vertices, say $r_1 < r_2 < \ldots < r_l$. It is likely that it is going to be much more difficult to study these types of graph. We shall attempt that analysis elsewhere.

**Acknowledgements**. The third author is partially supported by a research grant from the School of Sciences and the Vice Chancellor's office at his university.

## References


[1] N. Biggs. *Algebraic Graph Theory (2nd ed.)*, Cambridge Univ. Press, 1993.

[2] J.A. Bondy and U.S.R. Murthy. *Graph Theory with Applications*, American Elsevier Publishing Co., Inc, 1977.

[3] C.J. Colbourn, J.H. Dinitz. *CRC Handbook of Combinatorial Designs*, Boca Raton - FL, CRC Press, p. 652, 1996.

[4] C. Heuberger. On planarity and colorability of circulant graphs, *Discrete Math.* **268** (2003), no. 1-3, 153–169.

[5] D.A. Holton, J. Sheehan. *The Petersen Graph*, Cambridge - England, Cambridge University Press, 1993.