SOME p-ADIC CONGRUENCES FOR p^q-CATALAN NUMBERS

FLORIAN LUCA
Instituto de Matemáticas
Universidad Nacional Autónoma de México
C.P. 58089, Morelia, Michoacán, México
`fluca@matmor.unam.mx`

PAUL THOMAS YOUNG
Department of Mathematics
College of Charleston
Charleston, SC 29424, USA
`paul@math.cofc.edu`

June 14, 2010

Abstract

The integer $C_s(n) = \frac{1}{(s-1)n+1}\binom{sn}{n}$ is called the n-th s-Catalan number; when $s = 2$ we have $C_2(n) = c_n = \frac{1}{n+1}\binom{2n}{n}$, the usual Catalan number. In this paper, we look at some of the p-adic analytic properties of $C_{p^q}(n)$ for primes p. In particular, we show that the ratios $C_{p^q}(p^q n + 1)/C_{p^q}(n)$ may be p-adically interpolated using the p-adic Gamma function. Several congruences are derived from this representation, including a generalization of Wolstenholme's theorem.

1 Introduction

In a recent paper [2], we showed that among the integer sequence

$$c_n = \frac{1}{n+1}\binom{2n}{n}, \quad n \in \mathbb{Z}^+$$

of *Catalan numbers* c_n, the subsequence of odd Catalan numbers has a 2-adic limit, and it has the property that for all k, the first k odd Catalan numbers are distinct modulo 2^{k+1}. In this paper, we consider these questions p-adically for general primes p as they apply to sequences $C_{p^q}(n)$ of p^q-Catalan numbers. For any positive integer s the s-Catalan numbers $C_s(n)$ are defined by

$$C_s(n) = \frac{1}{(s-1)n+1}\binom{sn}{n}, \quad n \in \mathbb{Z}^+.$$

The first few values are $C_s(0) = C_s(1) = 1$, $C_s(2) = s$, $C_s(3) = s(3s-1)/2$, and we have $C_2(n) = c_n$ for all n. In [5] it was shown that for primes p the integer $C_{p^q}(n)$ is divisible by p in all cases except when $n = (p^{kq}-1)/(p^q-1)$ for some positive integer k, in which case $C_{p^q}(n) \equiv 1 \pmod{p^q}$. Here we give stronger versions of this congruence which may be regarded as generalizations of Wolstenholme's theorem.

Theorem 1. *Suppose $s = p^q$ is a power of a prime p and $n > 1$ is an integer such that the s-Catalan number $C_s(n)$ is not divisible by p. Then*

$$C_s(n) \equiv 1 \pmod{p^{q+r}}$$

where $r = 1$ if $p = 2$; $r = 2$ if $p = 3$; and $r = 3$ for $p \geqslant 5$. Furthermore, the above statement holds with $r = 4$ if and only if p is a Wolstenholme prime.

Wolstenholme's theorem [6] states that for primes $p \geqslant 5$ we have the congruence $\binom{2p}{p} \equiv 2 \pmod{p^3}$; there are many generalizations of this theorem, including those found in [3] and [7]. A *Wolstenholme prime* is a prime p for which $\binom{2p}{p} \equiv 2 \pmod{p^4}$, or equivalently for which p divides the numerator of the Bernoulli number B_{p-3}. To proceed further, let us recall some notation and terminology.

In what follows, $\mathbb{Z}_p$ denotes the ring of p-adic integers. We consider the p-adic Morita gamma function Γ_p defined for positive integers n by

$$\Gamma_p(n) = (-1)^n \prod_{\substack{0<j<n \\ p \nmid j}} j$$

(see [1], p. 368); it extends uniquely to a continuous function from $\mathbb{Z}_p$ to $\mathbb{Z}_p^\times$, and satisfies the translation functional equation

$$\Gamma_p(x+1) = \begin{cases} -x\Gamma_p(x), & x \in \mathbb{Z}_p^\times, \\ -\Gamma_p(x), & x \in p\mathbb{Z}_p. \end{cases} \tag{1}$$

Theorem 1 will be deduced from stronger congruences of Theorem 5 below. Those congruences will also be used to demonstrate the following result.

Theorem 2. *Suppose that $s = p^q$ is a power of a prime p; then the sequence of s-Catalan numbers which are not divisible by p converges p-adically to the limit*

$$\prod_{j=1}^{q} \Gamma_p\left(\frac{p^j}{1-p^q}\right) \prod_{i=1}^{\infty} \Gamma_p(p^i)^{-1}.$$

2 Proofs

Theorem 1 and Theorem 2 will be deduced from the following expression of certain ratios of s-Catalan numbers as ratios of binomial coefficients.

Proposition 3. *For all positive integers s and n we have*

$$\frac{C_s(sn+1)}{C_s(n)} = \frac{\binom{s^2n+s}{sn}}{\binom{sn+1}{n}}.$$

Proof. We have

$$\begin{aligned} \frac{C_s(sn+1)}{C_s(n)} &= \frac{(s-1)n+1}{(s-1)(sn+1)+1} \frac{\binom{s^2n+s}{sn+1}}{\binom{sn}{n}} = \frac{1}{s} \frac{\binom{s^2n+s}{sn+1}}{\binom{sn}{n}} \\ &= \frac{(s-1)sn+s}{s(sn+1)} \frac{\binom{s^2n+s}{sn}}{\binom{sn}{n}} = \frac{(s-1)n+1}{(sn+1)} \frac{\binom{s^2n+s}{sn}}{\binom{sn}{n}} = \frac{\binom{s^2n+s}{sn}}{\binom{sn+1}{n}}. \end{aligned}$$

□

From this proposition and Zhao's congruence ([7], Theorem 3.2), we may deduce that if $s = p^q$ is a power of a prime $p \geqslant 7$, we have

$$\frac{C_s(sn+1)}{C_s(n)} \equiv 1 + p^3 w_p n(sn+1)((s-1)n+1) \pmod{p^5\mathbb{Z}_p}, \tag{2}$$

where w_p (as in [7]) denotes the unique integer in $\{0, 1, ..., p^2 - 1\}$ such that $p^{-2}\sum_{k=1}^{p-1}\frac{1}{k} \equiv w_p \pmod{p^2\mathbb{Z}_p}$. To deduce the stronger congruences we have claimed we now express the ratios of binomial coefficients in Proposition 3 in terms of Γ_p as follows.

Corollary 4. *If $s = p^q$ is a power of a prime p, then*

$$\frac{C_s(sn+1)}{C_s(n)} = \prod_{j=1}^{q} \frac{\Gamma_p(p^{q+j}n + p^j)}{\Gamma_p(p^j n)\Gamma_p((p^q-1)p^j n + p^j)}.$$

Proof. We use the identity

$$\frac{\binom{pm}{pn}}{\binom{m}{n}} = \frac{\Gamma_p(pm)}{\Gamma_p(pn)\Gamma_p(p(m-n))} \tag{3}$$

(see [1], p. 382), to write the right side as a product of ratios of binomial coefficients; cancellation of common terms leaves $\binom{s^2n+s}{sn}/\binom{sn+1}{n}$, which equals the left side by Proposition 3. □

Corollary 4 above shows that the sequence of ratios $C_s(sn+1)/C_s(n)$ can be p-adically interpolated. We remark that since Γ_p is unit-valued, this corollary shows that the s-Catalan numbers $C_s(sn+1)$ and $C_s(n)$ always have the same p-adic valuation when $s = p^q$. The Jacobstahl-Kazandzidis congruences ([1], Cor. 11.6.22) state that the ratio of binomial coefficients in (3) is congruent to $K_p(m, n)$ modulo $p^4mn(m-n)\mathbb{Z}_p$, where

$$K_p(m,n) = \begin{cases} 1 - (B_{p-3}/3)p^3mn(m-n), & \text{if } p \geqslant 5, \\ 1 + 45mn(m-n), & \text{if } p = 3, \\ (-1)^{n(m-n)}P(m,n), & \text{if } p = 2; \end{cases} \tag{4}$$

here B_n denotes the nth Bernoulli number, and $P(m,n) = 1 + 6mn(m-n) - 4mn(m-n)(m^2 - mn + n^2) + 2(mn(m-n))^2$.

Stănică ([5], Lemma 5), showed that the p-adic valuation of the integer $C_{p^q}(n)$ is equal to $(S_p((p^q-1)n+1)-1)/(p-1)$, where $S_p(n)$ denotes the sum of the base p digits of n. It follows that $C_{p^q}(n)$ is not divisible by p if and only if $(p^q-1)n+1$ is a power of p, and therefore $n = (p^{kq}-1)/(p^q-1)$ for some integer k. The Jacobstahl-Kazandzidis congruences imply that the sequence of p-adic unit-valued p^q-Catalan numbers converges quite rapidly.

Theorem 5. *Suppose $s = p^q$ with p prime, and let $\{n_k\}_{k=0}^{\infty}$ be any sequence of nonnegative integers which satisfies the recurrence $n_{k+1} = sn_k + 1$. Then*

$$\frac{C_s(n_{k+1})}{C_s(n_k)} \equiv K_p(n_{k+1}, n_k) \pmod{p^{kq+4}\mathbb{Z}_p}.$$

Proof. We apply Corollary 4 with $n = n_k$, so that $sn + 1 = n_{k+1}$. This gives

$$\frac{C_s(n_{k+1})}{C_s(n_k)} = \prod_{j=1}^{q} \frac{\Gamma_p(p^j n_{k+1})}{\Gamma_p(p^j n_k)\Gamma_p(p^j(n_{k+1} - n_k))}. \tag{5}$$

The recurrence for $\{n_k\}_{k=0}^{\infty}$ implies that $n_{k+1} - n_k = s(n_k - n_{k-1})$, which shows that $n_{k+1} - n_k$ is a multiple of p^{kq}. By (4), the jth term in the product is congruent to $K_p(p^{j-1}n_{k+1}, p^{j-1}n_k)$ modulo $p^{kq+3j+4}\mathbb{Z}_p$ since $n_{k+1} - n_k$ is divisible by p^{kq}. Since $K_p(p^{j-1}n_{k+1}, p^{j-1}n_k) \equiv 1 \pmod{p^{kq+4}\mathbb{Z}_p}$ when $j > 1$, the desired result follows. □

The congruences of this theorem are stronger than those of (2) except for the $k = 0$ term of the sequence $\{n_k\}$, so (2) gives a stronger congruence that this theorem for values of n such that $n \not\equiv 1 \pmod{s}$. It is not hard to show that any sequence of nonnegative integers which satisfies the recurrence $n_{k+1} = sn_k + 1$ converges p-adically to the limit $(1 - s)^{-1}$. If we take $n_k = (p^{kq} - 1)/(p^q - 1)$, then the sequence $\{C_s(n_k)\}_{k=1}^{\infty}$ is precisely the sequence of p-adic unit p^q-Catalan numbers.

Corollary 6. *For $s = p^q$ with p prime, set $n_k = (p^{kq} - 1)/(p^q - 1)$. Then*

$$C_s(n_{k+1}) \equiv K_p(p^q + 1, 1) \pmod{p^{q+4}\mathbb{Z}_p}.$$

Proof. Apply Theorem 5 to $n_1, \ldots, n_k$, noting that $n_1 = 1$ and $n_2 = p^q + 1$. □

Corollary 6 above implies Theorem 1 by considering the various values of p. In particular, we observe that

$$K_2(2^q + 1, 1) \equiv 1 + 2^{q+1} \pmod{2^{2q+1}},$$

$$K_3(3^q + 1, 1) \equiv 1 + 5 \cdot 3^{q+2} \pmod{3^{2q+2}},$$

and

$$K_p(p^q + 1, 1) \equiv 1 + \frac{B_{p-3}}{3} \cdot p^{q+3} \pmod{p^{2q+3}\mathbb{Z}_p}$$

for $p \geqslant 5$. By the von Staudt - Clausen theorem ([1], Thm. 9.5.14), p does not divide the denominator of B_{p-3}, so $K_p(p^q+1,1) \equiv 1 \pmod{p^{q+3}\mathbb{Z}_p}$ for $p \geqslant 5$; and by definition $K_p(p^q+1,1) \equiv 1 \pmod{p^{q+4}\mathbb{Z}_p}$ if and only if p is a Wolstenholme prime. We remark in passing that the only known Wolstenholme primes are $p = 16843$ and $p = 2124679$.

By similarly evaluating $K_p(n_{k+1}, n_k)$ modulo $p^{kq+4}\mathbb{Z}_p$, we may use Theorem 5 to deduce the following result, as it was done in ([2], Theorem 3) for the case $s = 2$.

Corollary 7. *The first k odd 2^q-Catalan numbers are distinct modulo $2^{(k-1)q+2}$ but not modulo $2^{(k-1)q+1}$; the first k 3^q-Catalan numbers not divisible by 3 are distinct modulo $3^{(k-1)q+3}$ but not modulo $3^{(k-1)q+2}$; and if $p \geqslant 5$ is not a Wolstenholme prime, then the first k p^q-Catalan numbers not divisible by p are distinct modulo $p^{(k-1)q+4}$ but not modulo $p^{(k-1)q+3}$.*

Proof of Theorem 2. We take $n_k = (p^{kq}-1)/(p^q-1)$ in Theorem 5, and use equation (5) to write

$$\begin{aligned}\lim_{t\to\infty} C_s(n_t) &= \lim_{t\to\infty}\prod_{k=1}^{t-1}\frac{C_s(n_{k+1})}{C_s(n_k)} = \lim_{t\to\infty}\prod_{k=1}^{t-1}\prod_{j=1}^{q}\frac{\Gamma_p(p^j n_{k+1})}{\Gamma_p(p^j n_k)\Gamma_p(p^{kq+j})}\\ &= \lim_{t\to\infty}\prod_{j=1}^{q}\Gamma_p(p^j n_t)\prod_{i=1}^{tq}\Gamma_p(p^i)^{-1} = \prod_{j=1}^{q}\Gamma_p\left(\frac{p^j}{1-p^q}\right)\prod_{i=1}^{\infty}\Gamma_p(p^i)^{-1},\end{aligned}$$

by telescoping the above product and observing that $n_t \to (1-p^q)^{-1}$ as $t \to \infty$.

The same argument also shows that if $\{n_k\}_{k=0}^{\infty}$ is any sequence of nonnegative integers which satisfies the recurrence $n_{k+1} = sn_k + 1$, then

$$\lim_{k\to\infty} C_s(n_k) = C_s(n_0)\prod_{j=1}^{q}\Gamma_p\left(\frac{p^j}{1-p^q}\right)\prod_{j=1}^{q}\Gamma_p(p^j n_0)^{-1}\prod_{i=1}^{\infty}\Gamma_p(p^i(n_1-n_0))^{-1}.$$

Note that $n_1 - n_0$ is a positive integer congruent to 1 modulo $p^q - 1$, and $(S_p(n_1-n_0)-1)/(p-1)$ is the p-adic valuation of $C_s(n_k)$ for all k. The factor $\prod_{j=1}^{q}\Gamma_p(p^j n_0)$ in the above expression is an integer.

The factor $\prod_{j=1}^{q}\Gamma_p(p^j/(1-p^q))$ is algebraic, but this fact is somewhat nontrivial. One may use the translation functional equation (1) and reflection functional equation (see [1], Prop. 11.6.12) to rewrite this product as

$$\prod_{j=1}^{q}\Gamma_p\left(\frac{p^j}{1-p^q}\right) = (1-p^q)^{-1}\prod_{j=0}^{q-1}\Gamma_p\left(\frac{p^j}{p^q-1}\right)^{-1}.$$

Then the Gross-Koblitz formula (see [1], Theorem 11.7.5) says that the product of reciprocal Γ_p values on the right equals $(-p)^{1/(p-1)}$ divided by a Gauss sum for the multiplicative character $\omega^{-p^{q-1}}$, where ω is the Teichmüller character on the finite field of p^q elements. Therefore, this factor is algebraic. This implies that the limit of p-adic unit p^q-Catalan numbers differs from the limit of p-adic unit p^t-Catalan numbers by an algebraic factor.

We leave the determination of the transcendence or algebraicity of the factor $\prod_{i\geqslant 1}\Gamma_p(p^i)$ as an open problem to the reader. We remark that in ([4], Prop. 39.2), it is shown that

$$\prod_{i=1}^{\infty}\Gamma_p(p^i) = (-1)^{p-1}\lim_{n\to\infty}\frac{p^n!}{(-p)^{(p^n-1)/(p-1)}}.$$

Acknowledgements. This work started during a visit of F. L. at the Mathematics Department of the College of Charleston in September of 2008 and was finalized during a visit of P. T. Y. at the Mathematical Institute of the UNAM in Morelia, Mexico in November of 2009. Both authors thank the respective institutions for their hospitality and support.

References

[1] H. Cohen, Number Theory, *Graduate Texts in Mathematics* **239/240**, Springer-Verlag, New York, 2007.

[2] F. Luca and P.T. Young, 'On the binary expansion of the odd Catalan numbers'. *Applications of Fibonacci Numbers XIV*, to appear.

[3] R. Obláth, 'Congruences with binomial coefficients', *Proc. Math. Sci.* **1.6** (1934), 383-386.

[4] W. H. Schikhof, *Ultrametric calculus. An introduction to p-adic analysis*, Cambridge University Press, London, 1984.

[5] P. Stănică, 'p^q-Catalan numbers and squarefree binomial coefficients', *J. Number Theory* **100** (2003), 203-216.

[6] J. Wolstenholme, 'On certain properties of prime numbers', *Quart. J. Math.* **5** (1862), 3539.

[7] J. Zhao, ‘Bernoulli numbers, Wolstenholme’s theorem, and p^5 variations of Lucas’ theorem’, *J. Number Theory* **123** (2007), 18-26.