Some p-adic congruences for p^q-Catalan numbers

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Abstract

The integer $C_s(n) = \frac{1}{(s-1)n+1}\binom{sn}{n}$ is called the n-th s-Catalan number; when $s = 2$ we have $C_2(n) = c_n = \frac{1}{n+1}\binom{2n}{n}$, the usual Catalan number. In this paper, we look at some of the p-adic analytic properties of $C_{p^q}(n)$ for primes p. In particular, we show that the ratios $C_{p^q}(p^q n+1)/C_{p^q}(n)$ may be p-adically interpolated using the p-adic Gamma function. Several congruences are derived from this representation, including a generalization of Wolstenholme's theorem.

1 Introduction

In a recent paper [2], we showed that among the integer sequence

$$c_n = \frac{1}{n+1}\binom{2n}{n}, \quad n \in \mathbb{Z}^+$$

of *Catalan numbers* c_n, the subsequence of odd Catalan numbers has a 2-adic limit, and it has the property that for all k, the first k odd Catalan numbers are distinct modulo 2^{k+1}. In this paper, we consider these questions p-adically for general primes p as they apply to sequences $C_{p^q}(n)$ of p^q-Catalan numbers. For any positive integer s the s-Catalan numbers $C_s(n)$ are defined by

$$C_s(n) = \frac{1}{(s-1)n+1}\binom{sn}{n}, \quad n \in \mathbb{Z}^+.$$

The first few values are $C_s(0) = C_s(1) = 1$, $C_s(2) = s$, $C_s(3) = s(3s-1)/2$, and we have $C_2(n) = c_n$ for all n. In [5] it was shown that for primes p the integer $C_{p^q}(n)$ is divisible by p in all cases except when $n = (p^{kq}-1)/(p^q-1)$ for some positive integer k, in which case $C_{p^q}(n) \equiv 1 \pmod{p^q}$. Here we give stronger versions of this congruence which may be regarded as generalizations of Wolstenholme's theorem.

Theorem 1. *Suppose $s = p^q$ is a power of a prime p and $n > 1$ is an integer such that the s-Catalan number $C_s(n)$ is not divisible by p. Then*

$$C_s(n) \equiv 1 \pmod{p^{q+r}}$$

where $r = 1$ if $p = 2$; $r = 2$ if $p = 3$; and $r = 3$ for $p \geqslant 5$. Furthermore, the above statement holds with $r = 4$ if and only if p is a Wolstenholme prime.

Wolstenholme's theorem [6] states that for primes $p \geqslant 5$ we have the congruence $\binom{2p}{p} \equiv 2 \pmod{p^3}$; there are many generalizations of this theorem, including those found in [3] and [7]. A *Wolstenholme prime* is a prime p for which $\binom{2p}{p} \equiv 2 \pmod{p^4}$, or equivalently for which p divides the numerator of the Bernoulli number B_{p-3}. To proceed further, let us recall some notation and terminology.

In what follows, $\mathbb{Z}_p$ denotes the ring of p-adic integers. We consider the p-adic Morita gamma function Γ_p defined for positive integers n by

$$\Gamma_p(n) = (-1)^n \prod_{\substack{0<j<n \\ p \nmid j}} j$$

(see [1], p. 368); it extends uniquely to a continuous function from $\mathbb{Z}_p$ to $\mathbb{Z}_p^{\times}$, and satisfies the translation functional equation

$$\Gamma_p(x+1) = \begin{cases} -x\Gamma_p(x), & x \in \mathbb{Z}_p^{\times}, \\ -\Gamma_p(x), & x \in p\mathbb{Z}_p. \end{cases} \tag{1}$$

Theorem 1 will be deduced from stronger congruences of Theorem 5 below. Those congruences will also be used to demonstrate the following result.

Theorem 2. *Suppose that $s = p^q$ is a power of a prime p; then the sequence of s-Catalan numbers which are not divisible by p converges p-adically to the limit*

$$\prod_{j=1}^{q} \Gamma_p\left(\frac{p^j}{1-p^q}\right) \prod_{i=1}^{\infty} \Gamma_p(p^i)^{-1}.$$

2 Proofs

Theorem 1 and Theorem 2 will be deduced from the following expression of certain ratios of s-Catalan numbers as ratios of binomial coefficients.

Proposition 3. *For all positive integers s and n we have*

$$\frac{C_s(sn+1)}{C_s(n)} = \frac{\binom{s^2n+s}{sn}}{\binom{sn+1}{n}}.$$

Proof. We have

$$\begin{aligned} \frac{C_s(sn+1)}{C_s(n)} &= \frac{(s-1)n+1}{(s-1)(sn+1)+1} \frac{\binom{s^2n+s}{sn+1}}{\binom{sn}{n}} = \frac{1}{s} \frac{\binom{s^2n+s}{sn+1}}{\binom{sn}{n}} \\ &= \frac{(s-1)sn+s}{s(sn+1)} \frac{\binom{s^2n+s}{sn}}{\binom{sn}{n}} = \frac{(s-1)n+1}{(sn+1)} \frac{\binom{s^2n+s}{sn}}{\binom{sn}{n}} = \frac{\binom{s^2n+s}{sn}}{\binom{sn+1}{n}}. \end{aligned}$$

□

From this proposition and Zhao's congruence ([7], Theorem 3.2), we may deduce that if $s = p^q$ is a power of a prime $p \geqslant 7$, we have

$$\frac{C_s(sn+1)}{C_s(n)} \equiv 1 + p^3 w_p n(sn+1)((s-1)n+1) \pmod{p^5\mathbb{Z}_p}, \tag{2}$$

where w_p (as in [7]) denotes the unique integer in $\{0,1,...,p^2-1\}$ such that $p^{-2}\sum_{k=1}^{p-1}\frac{1}{k} \equiv w_p \pmod{p^2\mathbb{Z}_p}$. To deduce the stronger congruences we have claimed we now express the ratios of binomial coefficients in Proposition 3 in terms of Γ_p as follows.

Corollary 4. *If $s = p^q$ is a power of a prime p, then*

$$\frac{C_s(sn+1)}{C_s(n)} = \prod_{j=1}^{q} \frac{\Gamma_p(p^{q+j}n + p^j)}{\Gamma_p(p^j n)\Gamma_p((p^q-1)p^j n + p^j)}.$$

Proof. We use the identity

$$\frac{\binom{pm}{pn}}{\binom{m}{n}} = \frac{\Gamma_p(pm)}{\Gamma_p(pn)\Gamma_p(p(m-n))} \tag{3}$$

(see [1], p. 382), to write the right side as a product of ratios of binomial coefficients; cancellation of common terms leaves $\binom{s^2n+s}{sn}/\binom{sn+1}{n}$, which equals the left side by Proposition 3. □

Corollary 4 above shows that the sequence of ratios $C_s(sn+1)/C_s(n)$ can be p-adically interpolated. We remark that since Γ_p is unit-valued, this corollary shows that the s-Catalan numbers $C_s(sn+1)$ and $C_s(n)$ always have the same p-adic valuation when $s = p^q$. The Jacobstahl-Kazandzidis congruences ([1], Cor. 11.6.22) state that the ratio of binomial coefficients in (3) is congruent to $K_p(m,n)$ modulo $p^4mn(m-n)\mathbb{Z}_p$, where

$$K_p(m,n) = \begin{cases} 1-(B_{p-3}/3)p^3mn(m-n), & \text{if } p \geqslant 5, \\ 1+45mn(m-n), & \text{if } p=3, \\ (-1)^{n(m-n)}P(m,n), & \text{if } p=2; \end{cases} \tag{4}$$

here B_n denotes the nth Bernoulli number, and $P(m,n) = 1+6mn(m-n) - 4mn(m-n)(m^2-mn+n^2) + 2(mn(m-n))^2$.

Stănică ([5], Lemma 5), showed that the p-adic valuation of the integer $C_{p^q}(n)$ is equal to $(S_p((p^q-1)n+1)-1)/(p-1)$, where $S_p(n)$ denotes the sum of the base p digits of n. It follows that $C_{p^q}(n)$ is not divisible by p if and only if $(p^q-1)n+1$ is a power of p, and therefore $n = (p^{kq}-1)/(p^q-1)$ for some integer k. The Jacobstahl-Kazandzidis congruences imply that the sequence of p-adic unit-valued p^q-Catalan numbers converges quite rapidly.

Theorem 5. *Suppose $s = p^q$ with p prime, and let $\{n_k\}_{k=0}^{\infty}$ be any sequence of nonnegative integers which satisfies the recurrence $n_{k+1} = sn_k + 1$. Then*

$$\frac{C_s(n_{k+1})}{C_s(n_k)} \equiv K_p(n_{k+1}, n_k) \pmod{p^{kq+4}\mathbb{Z}_p}.$$

Proof. We apply Corollary 4 with $n = n_k$, so that $sn + 1 = n_{k+1}$. This gives

$$\frac{C_s(n_{k+1})}{C_s(n_k)} = \prod_{j=1}^{q} \frac{\Gamma_p(p^j n_{k+1})}{\Gamma_p(p^j n_k)\Gamma_p(p^j(n_{k+1} - n_k))}. \tag{5}$$

The recurrence for $\{n_k\}_{k=0}^{\infty}$ implies that $n_{k+1} - n_k = s(n_k - n_{k-1})$, which shows that $n_{k+1} - n_k$ is a multiple of p^{kq}. By (4), the jth term in the product is congruent to $K_p(p^{j-1}n_{k+1}, p^{j-1}n_k)$ modulo $p^{kq+3j+4}\mathbb{Z}_p$ since $n_{k+1} - n_k$ is divisible by p^{kq}. Since $K_p(p^{j-1}n_{k+1}, p^{j-1}n_k) \equiv 1 \pmod{p^{kq+4}\mathbb{Z}_p}$ when $j > 1$, the desired result follows. □

The congruences of this theorem are stronger than those of (2) except for the $k = 0$ term of the sequence $\{n_k\}$, so (2) gives a stronger congruence that this theorem for values of n such that $n \not\equiv 1 \pmod{s}$. It is not hard to show that any sequence of nonnegative integers which satisfies the recurrence $n_{k+1} = sn_k + 1$ converges p-adically to the limit $(1 - s)^{-1}$. If we take $n_k = (p^{kq} - 1)/(p^q - 1)$, then the sequence $\{C_s(n_k)\}_{k=1}^{\infty}$ is precisely the sequence of p-adic unit p^q-Catalan numbers.

Corollary 6. *For $s = p^q$ with p prime, set $n_k = (p^{kq} - 1)/(p^q - 1)$. Then*

$$C_s(n_{k+1}) \equiv K_p(p^q + 1, 1) \pmod{p^{q+4}\mathbb{Z}_p}.$$

Proof. Apply Theorem 5 to $n_1, \ldots, n_k$, noting that $n_1 = 1$ and $n_2 = p^q + 1$. □

Corollary 6 above implies Theorem 1 by considering the various values of p. In particular, we observe that

$$K_2(2^q + 1, 1) \equiv 1 + 2^{q+1} \pmod{2^{2q+1}},$$

$$K_3(3^q + 1, 1) \equiv 1 + 5 \cdot 3^{q+2} \pmod{3^{2q+2}},$$

and

$$K_p(p^q + 1, 1) \equiv 1 + \frac{B_{p-3}}{3} \cdot p^{q+3} \pmod{p^{2q+3}\mathbb{Z}_p}$$

for $p \geqslant 5$. By the von Staudt - Clausen theorem ([1], Thm. 9.5.14), p does not divide the denominator of B_{p-3}, so $K_p(p^q+1,1) \equiv 1 \pmod{p^{q+3}\mathbb{Z}_p}$ for $p \geqslant 5$; and by definition $K_p(p^q+1,1) \equiv 1 \pmod{p^{q+4}\mathbb{Z}_p}$ if and only if p is a Wolstenholme prime. We remark in passing that the only known Wolstenholme primes are $p = 16843$ and $p = 2124679$.

By similarly evaluating $K_p(n_{k+1}, n_k)$ modulo $p^{kq+4}\mathbb{Z}_p$, we may use Theorem 5 to deduce the following result, as it was done in ([2], Theorem 3) for the case $s = 2$.

Corollary 7. *The first k odd 2^q-Catalan numbers are distinct modulo $2^{(k-1)q+2}$ but not modulo $2^{(k-1)q+1}$; the first k 3^q-Catalan numbers not divisible by 3 are distinct modulo $3^{(k-1)q+3}$ but not modulo $3^{(k-1)q+2}$; and if $p \geqslant 5$ is not a Wolstenholme prime, then the first k p^q-Catalan numbers not divisible by p are distinct modulo $p^{(k-1)q+4}$ but not modulo $p^{(k-1)q+3}$.*

Proof of Theorem 2. We take $n_k = (p^{kq}-1)/(p^q-1)$ in Theorem 5, and use equation (5) to write

$$\begin{aligned}\lim_{t\to\infty} C_s(n_t) &= \lim_{t\to\infty}\prod_{k=1}^{t-1}\frac{C_s(n_{k+1})}{C_s(n_k)} = \lim_{t\to\infty}\prod_{k=1}^{t-1}\prod_{j=1}^{q}\frac{\Gamma_p(p^j n_{k+1})}{\Gamma_p(p^j n_k)\Gamma_p(p^{kq+j})}\\ &= \lim_{t\to\infty}\prod_{j=1}^{q}\Gamma_p(p^j n_t)\prod_{i=1}^{tq}\Gamma_p(p^i)^{-1} = \prod_{j=1}^{q}\Gamma_p\left(\frac{p^j}{1-p^q}\right)\prod_{i=1}^{\infty}\Gamma_p(p^i)^{-1},\end{aligned}$$

by telescoping the above product and observing that $n_t \to (1-p^q)^{-1}$ as $t \to \infty$.

The same argument also shows that if $\{n_k\}_{k=0}^{\infty}$ is any sequence of nonnegative integers which satisfies the recurrence $n_{k+1} = sn_k + 1$, then

$$\lim_{k\to\infty} C_s(n_k) = C_s(n_0)\prod_{j=1}^{q}\Gamma_p\left(\frac{p^j}{1-p^q}\right)\prod_{j=1}^{q}\Gamma_p(p^j n_0)^{-1}\prod_{i=1}^{\infty}\Gamma_p(p^i(n_1-n_0))^{-1}.$$

Note that $n_1 - n_0$ is a positive integer congruent to 1 modulo $p^q - 1$, and $(S_p(n_1-n_0)-1)/(p-1)$ is the p-adic valuation of $C_s(n_k)$ for all k. The factor $\prod_{j=1}^{q}\Gamma_p(p^j n_0)$ in the above expression is an integer.

The factor $\prod_{j=1}^{q}\Gamma_p(p^j/(1-p^q))$ is algebraic, but this fact is somewhat nontrivial. One may use the translation functional equation (1) and reflection functional equation (see [1], Prop. 11.6.12) to rewrite this product as

$$\prod_{j=1}^{q}\Gamma_p\left(\frac{p^j}{1-p^q}\right) = (1-p^q)^{-1}\prod_{j=0}^{q-1}\Gamma_p\left(\frac{p^j}{p^q-1}\right)^{-1}.$$

Then the Gross-Koblitz formula (see [1], Theorem 11.7.5) says that the product of reciprocal Γ_p values on the right equals $(-p)^{1/(p-1)}$ divided by a Gauss sum for the multiplicative character $\omega^{-p^{q-1}}$, where ω is the Teichmüller character on the finite field of p^q elements. Therefore, this factor is algebraic. This implies that the limit of p-adic unit p^q-Catalan numbers differs from the limit of p-adic unit p^t-Catalan numbers by an algebraic factor.

We leave the determination of the transcendence or algebraicity of the factor $\prod_{i\geqslant 1}\Gamma_p(p^i)$ as an open problem to the reader. We remark that in ([4], Prop. 39.2), it is shown that

$$\prod_{i=1}^{\infty}\Gamma_p(p^i) = (-1)^{p-1}\lim_{n\to\infty}\frac{p^n!}{(-p)^{(p^n-1)/(p-1)}}.$$

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