EXPLICIT COMPUTATION OF GROSS-STARK UNITS OVER REAL QUADRATIC FIELDS

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Dedicated to the memory of David R. Hayes

Abstract. We present an effective and practical algorithm for computing Gross-Stark units over a real quadratic base field F. Our algorithm allows us to explicitly construct certain relative abelian extensions of F where these units lie, using only information from the base field. These units were recently proved to always exist within the correct extension fields of F by Dasgupta, Darmon, and Pollack, without directly producing them.

1. Introduction

In 1981, Benedict Gross [Gr] proposed a refined conjecture concerning the values of the first derivatives of certain p-adic partial zeta functions at $s = 0$ which, if correct, allows one to explicitly construct relative abelian extensions of totally real algebraic number fields in the spirit of Hilbert's 12th problem via p-adic analytic functions, as opposed to complex-valued such functions. A major breakthrough concerning this conjecture was very recently published in a seminal article by Dasgupta, Darmon, and Pollack [DDP]. They obtain a conditional proof of Gross's conjecture subject essentially only to the hypothesis that Leopoldt's conjecture holds for the totally real base field. In this paper, we study a version of Gross's conjecture to which the proof in [DDP] applies unconditionally, namely, the base field F is real quadratic and the prime p splits completely in F. This allows us to present an effective and practical algorithm for constructing certain abelian extensions over a real quadratic field F that is guaranteed to succeed based upon the results in [DDP]. This solves a problem of long-standing algorithmic interest and provides further important information at the same time. As pointed out in [DDP] (see Remark 8 on p. 443), their approach does not construct the Gross-Stark unit at the center of Gross's conjecture in a direct fashion. In [Gr], Gross presented a proof of his conjecture for abelian extensions over $\mathbb{Q}$ as well as an explicit formula for the unit in question in terms of certain Gauss sums which are related to special values of Morita's p-adic gamma function via the Gross-Koblitz formula (see also [P] for a new approach to this circle of ideas). Although it is still unknown whether analogous *algebraic* formulas exist for expressing the relevant Gross-Stark units over totally real base fields other than $\mathbb{Q}$, the p-adic version of (12) and (14) in Section 3 may be viewed as providing an analogous *analytic* formula in the case of real

Date: April 27th, 2012.

2000 *Mathematics Subject Classification.* Primary 11S40; Secondary 11R11, 11R37, 11R42, 11Y40.

Key words and phrases. Gross's conjecture; Gross-Stark units; p-adic Multiple Zeta and Log Gamma Functions.

quadratic base fields—the role played by the Diamond function G_p in Gross's formula (see Eq. (6) below) is played in our formula by the p-adic log double gamma function $G_{p,2}$ developed in [TY]. The leading goal of our algorithmic approach is to obtain directly the Gross-Stark unit itself by first constructing a polynomial with coefficients in the base field of which it is a root. With this polynomial in hand, one has the means to produce the specific abelian extension of the base field where the Gross-Stark unit lies as well as being able to provide both clues and evidence towards a potential algebraic formula expressing these units over base fields other than $\mathbb{Q}$.

The main difficulty, from an algorithmic standpoint, in obtaining explicitly the Gross-Stark units over totally real fields other than $\mathbb{Q}$ is the computation of the first derivatives of the p-adic partial zeta functions at $s = 0$. When the base field F is real quadratic, there is an especially nice way to handle this computation based upon a continued fraction algorithm due to Zagier [Z] and Hayes [H] (see Section 3). For this reason, we restrict ourselves to only giving a detailed algorithm when F is real quadratic, however, we wish to emphasize that similar methods to those in Section 3 may be employed over higher degree totally real base fields as well. An alternate method, using different formulas, to carry out these computations was recently presented by Kashio and Yoshida [KY1], [KY2]. Another conjecture, building upon Gross's original conjecture, was recently proposed by Darmon and Dasgupta [DD] and also numerically studied by Dasgupta [Ds] over a real quadratic base field F. In their conjecture, p is assumed to remain inert in F, complementing nicely the situation considered here.

Let $\mathbb{Z}$, $\mathbb{Z}^+$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{R}^+$, $\mathbb{C}$, and $\mathbb{I}_m$ denote the set of rational integers, positive integers, rational numbers, real numbers, positive real numbers, complex numbers, and $\mathbb{Z}/m\mathbb{Z}$ for a fixed integer $m \geq 2$, respectively. If R is a ring with multiplicative identity $1 \neq 0$, then $R^\times$ denotes the set of units in R. Let $\overline{\mathbb{Q}}$ denote a fixed algebraic closure of $\mathbb{Q}$, considered abstractly as opposed to being considered as a subfield of $\mathbb{C}$. If X is a finite set, then $|X|$ denotes its cardinality.

The conjecture of Gross on which we focus our attention throughout is closely related to the Brumer-Stark conjecture and the set-up for both is the same: K/F is a relative abelian extension of number fields with K totally complex and F totally real. Of necessity, $2 \mid n = [\mathsf{K} : \mathsf{F}]$. These conjectures, in common with all conjectures of Stark-type, predict a precise match-up between algebraic data on one side of the equation and analytic data on the other side. If $G = \mathrm{Gal}(\mathsf{K}/\mathsf{F})$, then for each automorphism $\sigma \in G$ there is a partial zeta function $\zeta_T(s, \sigma)$ (T is a finite set of primes in F whose precise definition will be given in Section 2) whose value $\zeta_T(0, \sigma)$ at $s = 0$ is a rational number. The rationality of these values was first proved by Klingen [Kl] and Siegel [Si]. If w_{K} denotes the number of roots of unity in K, a general theorem due to Barsky [Ba], Cassou-Noguès [CN], and Deligne-Ribet [DR] states that $w_{\mathsf{K}}\zeta_T(0, \sigma) \in \mathbb{Z}$. These integers, $w_{\mathsf{K}}\zeta_T(0, \sigma)$, $\sigma \in G$, constitute the analytic data that goes into the Brumer-Stark conjecture. The conjecture of Gross, which is a p-adic refinement of the Brumer-Stark conjecture, requires the rational numbers $\zeta_T(0, \sigma)$, $\sigma \in G$, as well as the first derivatives evaluated at $s = 0$, $\zeta'_{S,p}(0, \sigma)$ (the set S is similar to T above), of certain p-adic partial zeta functions, one defined for each $\sigma \in G$. The algebraic data in each case is of the same type: A well-defined algebraic number lying in the field K, denoted by $\alpha_{\mathsf{B\text{-}S}}$ and α_{Gr}, respectively. We will later see that these two numbers are closely related and we generically call these α's "Stark units". In each conjecture, we will focus on a nonzero prime ideal $\mathfrak{p}$ of the ring of integers $\mathcal{O}_{\mathsf{F}}$ of F that splits completely in the extension K/F and which lies above the prime $p \in \mathbb{Z}$ used to define the p-adic partial zeta functions mentioned above. The α's lie in the subset $U_{\mathfrak{p}}$ of K defined by

$$U_{\mathfrak{p}} = \{\beta \in \mathsf{K}^{\times} \ : \ |\beta|_{\mathfrak{Q}} = 1 \quad \text{if } \mathfrak{Q} \text{ does not divide } \mathfrak{p}\}, \tag{1}$$

and in particular the absolute values of the α's with respect to every complex embedding $\mathsf{K} \hookrightarrow \mathbb{C}$ are all equal to one. The only nontrivial absolute values associated to a given α arise from the $n = |G|$ distinct prime ideals $\mathfrak{P} \subset \mathcal{O}_{\mathsf{K}}$ lying over $\mathfrak{p}$ (remember that $\mathfrak{p}$ splits completely!) and these values are specified by the analytic data mentioned above: The n values $\zeta_T(0,\sigma)$, $\sigma \in G$.

As an introduction to the Brumer-Stark and Gross conjectures, we work through a detailed example over the base field $\mathbb{Q}$ (for ease of presentation, all number fields in this example are considered as subfields of $\mathbb{C}$). The computations we carry out over real quadratic base fields follow a similar though more intricate pattern. Let L denote the field of 16th roots of unity. The Galois group of the extension $\mathsf{L}/\mathbb{Q}$ is isomorphic to $(\mathbb{Z}/16\mathbb{Z})^{\times} \cong \mathbb{I}_4 \times \mathbb{I}_2$, which in turn is isomorphic to the ray class group modulo $(16)p_\infty$, where p_∞ denotes the unique infinite place of $\mathbb{Q}$. Consider the following Dirichlet character χ defined modulo 16: $\chi(1) = 1,\ \chi(3) = i,\ \chi(5) = i,\ \chi(7) = 1,\ \chi(9) = -1,\ \chi(11) = -i,\ \chi(13) = -i,\ \chi(15) = -1$, and $\chi(2n) = 0$ for all $n \in \mathbb{Z}$. Note that χ is an odd quartic character and corresponding to the cyclic group $\langle\chi\rangle$ generated by χ is an intermediate field K with $\mathbb{Q} \subset \mathsf{K} \subset \mathsf{L}$ (cf. [Wa], Chapter 3). The field K is totally complex since χ is odd and $G := \mathrm{Gal}(\mathsf{K}/\mathbb{Q}) \cong \mathbb{I}_4$. We have $\mathsf{K} = \mathbb{Q}(\mathrm{Tr}_{\mathsf{L}/\mathsf{K}}(\zeta_{16})) = \mathbb{Q}(\zeta_{16} + \zeta_{16}^7) = \mathbb{Q}(\theta)$, where $\zeta_{16} = \exp(2\pi i/16)$ and θ satisfies the irreducible polynomial $x^4 + 4x^2 + 2$. Let T denote the set $\{p_\infty, 2\}$ of all places in $\mathbb{Q}$ that ramify in $\mathsf{K}/\mathbb{Q}$ and we consider the Brumer-Stark and Gross conjectures with respect to the extension $\mathsf{K}/\mathbb{Q}$. A crucial ingredient, particularly in Gross's conjecture, is the specification of a finite prime in the base field that splits completely in the relative abelian extension of interest. Since $\chi(7) = 1$, the prime $\mathfrak{p} = (7)$ splits completely in K and we choose this prime as our distinguished split prime and let $S = T \cup \{\mathfrak{p}\}$. The partial zeta function associated to the identity automorphism $\sigma_0 \in G$ is given by

$$\zeta_T(s,\sigma_0) = \zeta_1(s,1,16) + \zeta_1(s,7,16),$$

where $\zeta_1(s,x,f) = \sum_{n=0}^{\infty}(x+nf)^{-s}$ (we assume $x, f \in \mathbb{R}^+$ in this definition). After meromorphic continuation, we have $\zeta_1(0,x,f) = (\frac{1}{2} - \frac{x}{f})$, and so $\zeta_T(0,\sigma_0) = 1/2$. The prime 3 is inert in K since $\chi(3) = i$ and therefore $G = \langle\sigma\rangle$, where σ is the Frobenius automorphism of 3. We have $\zeta_T(s,\sigma) = \zeta_1(s,3,16) + \zeta_1(s,5,16)$ and $\zeta_T(0,\sigma) = 1/2$. Similarly, $\zeta_T(0,\sigma^2) = \zeta_T(0,\sigma^3) = -1/2$. For this example, $w_{\mathsf{K}} = 2$. If $\mathfrak{P} \subset \mathcal{O}_{\mathsf{K}}$ is a fixed prime ideal lying over $\mathfrak{p}$, the Brumer-Stark conjecture specifies precisely the $\mathfrak{P}$-adic ordinal of the Galois conjugates of an element $\alpha_{\mathsf{B\text{-}S}} \in U_{\mathfrak{p}}$, whose existence it predicts, as follows:

$$\mathrm{ord}_{\mathfrak{P}}(\sigma(\alpha_{\mathsf{B\text{-}S}})) = w_{\mathsf{K}}\zeta_T(0,\sigma) \quad \text{for all } \sigma \in G. \tag{2}$$

Note that the absolute value of $\alpha_{\mathsf{B\text{-}S}}$ is specified at *all* places of K and therefore $\alpha_{\mathsf{B\text{-}S}}$, assuming it exists, is uniquely defined up to a root of unity in K once $\mathfrak{P}$ is fixed. The Brumer-Stark conjecture is usually stated in a more global fashion (see [RT], for example), but we have stated it here with respect to a completely split finite prime $\mathfrak{p}$ in order to draw a more direct connection with the conjecture of Gross. The element $\alpha_{\mathsf{B\text{-}S}}$ is also predicted to satisfy an additional "abelian condition" which we will return to in Section 2. The existence of $\alpha_{\mathsf{B\text{-}S}}$ implies a relation among the ideal classes in K containing the prime ideals above $\mathfrak{p}$ which is reminiscent of Stickelberger's theorem on the factorization of Gauss sums in cyclotomic extensions of $\mathbb{Q}$. Indeed, the Brumer-Stark conjecture has been proven for totally complex abelian extensions of $\mathbb{Q}$ by use of Stickelberger's theorem and $\alpha_{\mathsf{B\text{-}S}}$ may be expressed in this case in terms of a normalized Gauss sum raised to a specific power (see [Ta], p. 109).

The special values $\zeta_T(0,\sigma)$, $\sigma \in G$, are computed working in the ray class group modulo $(16)p_\infty$ and to compute the values $\zeta'_{S,7}(0,\sigma)$, $\sigma \in G$, we work modulo $(16 \cdot 7)p_\infty$. In order to motivate our p-adic computations, we first recall that as a complex-valued function we have

$$\zeta_1'(0,x,f) = \left(\frac{x}{f} - \frac{1}{2}\right)\log f + \log\left\{\frac{\Gamma\left(\frac{x}{f}\right)}{\sqrt{2\pi}}\right\}, \tag{3}$$

where $\Gamma(s)$ is the classical gamma function. The function $\zeta_S(s,\sigma)$ differs from $\zeta_T(s,\sigma)$ by an Euler factor, namely, $\zeta_S(s,\sigma) = (1 - 1/7^s)\zeta_T(s,\sigma)$, and $\zeta'_S(0,\sigma)$ may be computed by adding together a finite number of expressions of the form appearing on the right side of (3). For example, with respect to the identity automorphism σ_0 we define the set $J = \{1, 17, 23, 33, 39, 55, \ldots, 103\}$ of all positive integers less than $f = 112 = 16 \cdot 7$, congruent to 1 or 7 modulo 16, and not divisible by 7, and find that

$$\zeta'_S(0,\sigma_0) = \sum_{x \in J} \log\left\{\frac{\Gamma\left(\frac{x}{112}\right)}{\sqrt{2\pi}}\right\} \tag{4}$$

(the terms involving $(\frac{x}{f} - \frac{1}{2})\log f$ all cancel). An important result due to Kashio ([K], Theorem 6.2) says in this case that the value $\zeta'_{S,7}(0,\sigma_0)$ is given by the same formula as in (4) once the "correct" p-adic interpretation is given to the log gamma function. The expression

$$\left(\frac{x}{f} - \frac{1}{2}\right)\log\left(\frac{x}{f}\right) - \left(\frac{x}{f}\right) + \sum_{j=2}^{\infty} \frac{(-1)^j (j-2)!}{j!} B_j \left(\frac{x}{f}\right)^{1-j} \tag{5}$$

is the asymptotic expansion (Stirling's series) of $\log(\Gamma(x/f)/\sqrt{2\pi})$ (B_j is the jth Bernoulli number). The infinite sum in (5) does not converge in $\mathbb{C}$ but Diamond [D] proved that it *does* converge p-adically if $|\frac{x}{f}|_p > 1$. Replacing log by the Iwasawa p-adic logarithm in (5), we let $G_p(\frac{x}{f})$ denote the value in the p-adic rationals $\mathbb{Q}_p$ equal to the expression in (5) when $\frac{x}{f} \in \mathbb{Q}$ and $|\frac{x}{f}|_p > 1$. By Kashio's result and a straightforward computation, we obtain (compare the first equation in (6) below with (4.3) in [Gr])

$$\zeta'_{S,7}(0,\sigma_0) = \sum_{x \in J} G_7(x/112) = 2 \cdot 7 + 4 \cdot 7^2 + 5 \cdot 7^4 + \cdots \tag{6}$$

and it should be noted that (5) offers not only an elegant but also an efficient formula for computing $G_p(\frac{x}{f})$. In order to state Gross's conjecture, we fix a prime ideal $\mathfrak{P} \subset \mathcal{O}_{\mathsf{K}}$ over $\mathfrak{p} = (7)$ which in this example corresponds to an embedding $\mathsf{K} \hookrightarrow \mathbb{Q}_p$ with $p = 7$ which we denote elementwise for all $x \in \mathsf{K}$ by $x \mapsto x_{\mathfrak{P}} \in \mathbb{Q}_p$.

The conjecture of Gross states that there exists an element $\alpha_{\mathsf{Gr}} \in U_{\mathfrak{p}}$ such that

$$(\sigma(\alpha_{\mathsf{Gr}}))_{\mathfrak{P}} = p^{w_p \zeta_T(0,\sigma)} \cdot \exp_p(-w_p \zeta'_{S,p}(0,\sigma)) \quad \text{for all } \sigma \in G, \tag{7}$$

where $w_p = 6$ is the number of roots of unity in $\mathbb{Q}_7$. Note that α_{Gr}, assuming it exists, is uniquely defined once $\mathfrak{P}$ is fixed and is equal (see [Ta], p. 136) to $\alpha_{\mathsf{B\text{-}S}}^m$ multiplied by a root of unity in K, where $w_p = m \cdot w_{\mathsf{K}}$ ($w_{\mathsf{K}} \mid w_p$ by the existence of the embedding $\mathsf{K} \hookrightarrow \mathbb{Q}_p$). This is consistent with the Brumer-Stark Eq. (2) since the p-adic exponential function on the right side of (7) takes its values in $1 + p\mathbb{Z}_p$, with $\mathbb{Z}_p$ denoting the set of p-adic integers. The appearance of an expression of the form $\exp(-w\zeta'(0,\sigma))$ on the right side of (7) directly corresponds to the "classic" statement of Stark's conjecture ([T], p. 300) where the distinguished completely split prime is an infinite prime, and thus Gross's conjecture is really a fusion of the Brumer-Stark and classic Stark conjectures. And just as for the classic Stark conjecture, Gross's analytic expression on the right side of (7) may be used to compute a polynomial with coefficients in the base field satisfied by the Gross-Stark unit α_{Gr}, using *only* information from the base field, enabling us to explicitly construct the relative abelian extension K where α_{Gr} lies in the spirit of Hilbert's 12th problem. In the present example, the polynomial $x^4 + \frac{964}{7^3}x^3 + \frac{461350}{7^6}x^2 + \frac{964}{7^3}x + 1$ that α_{Gr} satisfies is easily obtained from the 4 Galois conjugates of α_{Gr} as given by Eq. (7). This clearly demonstrates the added strength of Gross's conjecture versus that of Brumer-Stark; the field K must first be explicitly known *before* computing $\alpha_{\mathsf{B\text{-}S}}$ (cf. [RT], [GRT]). Gross [Gr] proved his conjecture for all complex abelian extensions of $\mathbb{Q}$ using the Gross-Koblitz formula, which itself is a p-adic refinement of the theorem of Stickelberger referred to earlier in connection to the Brumer-Stark conjecture over $\mathbb{Q}$, and expresses Gauss sums in terms of Morita's p-adic gamma function.

2. Gross's Conjecture over a real quadratic field

In this section, we consider Gross's refined conjecture over a real quadratic field and refer to §3 of [Gr], Chap. VI, §4 of [Ta], or [DDP], for the more general statement of the conjecture. Let $\mathsf{F} \subset \overline{\mathbb{Q}}$ be a fixed real quadratic field having discriminant $d_{\mathsf{F}} > 0$ and set

$$f[d_{\mathsf{F}}] = \begin{cases} x^2 - d_{\mathsf{F}}/4 & \text{if} \quad d_{\mathsf{F}} \equiv 0 \bmod (4), \\ x^2 - x - (d_{\mathsf{F}} - 1)/4 & \text{if} \quad d_{\mathsf{F}} \equiv 1 \bmod (4). \end{cases}$$

If $\theta \in \overline{\mathbb{Q}}$ is a root of $f[d_{\mathsf{F}}]$, then $\mathsf{F} = \mathbb{Q}(\theta)$ and $\mathcal{O}_{\mathsf{F}} = [1, \theta]$, where $[\alpha, \beta] := \{a\alpha + b\beta \mid a, b \in \mathbb{Z}\}$. All ideals we consider, whether fractional, integral, or prime, are always understood to be nonzero. If $\mathfrak{a} \subseteq \mathcal{O}_{\mathsf{F}}$ is an integral ideal, let $\mathrm{N}\mathfrak{a} = [\mathcal{O}_{\mathsf{F}} : \mathfrak{a}]$ denote its norm. A rational prime is always assumed to be positive by default. Let $\theta^{(1)}$ denote the positive real root of $f[d_{\mathsf{F}}]$ and $\theta^{(2)}$ the negative real root. The two real embeddings of F into $\mathbb{R}$ are specified in the following order:

$$\begin{aligned} i_1 : \mathsf{F} \hookrightarrow \mathbb{R} \quad &\text{is defined by the map} \quad a + b\theta \mapsto a + b\theta^{(1)}, \ (a, b \in \mathbb{Q}), \\ i_2 : \mathsf{F} \hookrightarrow \mathbb{R} \quad &\text{is defined by the map} \quad a + b\theta \mapsto a + b\theta^{(2)}. \end{aligned}$$

The two infinite primes corresponding to the two real embeddings of F are denoted by $\mathfrak{p}_\infty^{(1)}$ and $\mathfrak{p}_\infty^{(2)}$, respectively. Let p be a fixed rational prime that splits completely in F with $(p) = \mathfrak{p}\overline{\mathfrak{p}}$, $\mathfrak{p} \neq \overline{\mathfrak{p}}$. We assume that K is a totally complex algebraic number field, relatively abelian over F, and we further assume that $\mathfrak{p}$ splits completely in the extension K/F. Let S be a finite set of places of F containing $\mathfrak{p}_\infty^{(1)}$, $\mathfrak{p}_\infty^{(2)}$, $\overline{\mathfrak{p}}$, $\mathfrak{p}$, as well as any finite primes which ramify in K. The set $T \subsetneq S$ is chosen in such a way that $S = T \cup \{\mathfrak{p}\}$ and therefore T contains all primes that ramify in the extension K/F. If $G = \mathrm{Gal}(\mathsf{K}/\mathsf{F})$, for each $\sigma \in G$ we define

$$\zeta_T(s,\sigma) = \sum_{\sigma_{\mathfrak{a}}=\sigma} \frac{1}{\mathrm{N}\mathfrak{a}^s}, \tag{8}$$

where the sum is taken over all integral ideals $\mathfrak{a} \subseteq \mathcal{O}_{\mathsf{F}}$ not divisible by any finite prime in T and having the same Artin symbol $(\mathsf{K}/\mathsf{F},\mathfrak{a}) = \sigma_{\mathfrak{a}} = \sigma$. The infinite sum on the right side of (8) converges only for $\Re(s) > 1$, but $\zeta_T(s,\sigma)$ has a meromorphic continuation to all of $\mathbb{C}$ with exactly one (simple) pole at $s = 1$. Let w_p denote the number of roots of unity in $\mathbb{Q}_p$, namely, $w_2 = 2$ and $w_p = p - 1$ if p is odd. As mentioned in the Introduction, $w_p\zeta_T(0,\sigma) \in \mathbb{Z}$ for all $\sigma \in G$. The partial zeta functions $\zeta_S(s,\sigma)$, $\sigma \in G$, are defined exactly as $\zeta_T(s,\sigma)$ above. Since both prime ideals in $\mathcal{O}_{\mathsf{F}}$ lying over p are in S, a p-adic partial zeta function $\zeta_{S,p}(s,\sigma)$ exists for each $\sigma \in G$ whose values interpolate exactly those of $\zeta_S(s,\sigma)$ at certain nonpositive integers (see [CN], Corollaire 23) and $\zeta_{S,p}(s,\sigma)$ is p-adically differentiable at $s = 0$.

Let $\mathfrak{P} \subset \mathcal{O}_{\mathsf{K}}$ be a fixed prime ideal lying over $\mathfrak{p}$ and let $x_{\mathfrak{P}}$ denote the image of $x \in \mathsf{K}$ with respect to the embedding $\mathsf{K} \hookrightarrow \mathbb{Q}_p$ corresponding to $\mathfrak{P}$. Given the above notations and assumptions, we can now state

Gross's Refined Conjecture ([Gr]). *There exists a unique element $\alpha_{\mathsf{Gr}} \in U_{\mathfrak{p}} \subset \mathsf{K}^\times$ such that*

(1) $(\sigma(\alpha_{\mathsf{Gr}}))_{\mathfrak{P}} = p^{w_p\zeta_T(0,\sigma)} \cdot \exp_p(-w_p\zeta'_{S,p}(0,\sigma))$ *for all $\sigma \in G$, and*

(2) $\mathsf{K}(\alpha_{\mathsf{Gr}}^{1/w_{\mathsf{K}}})$ *is an abelian extension of* F.

Part 2 of this conjecture, the so-called "abelian condition", is an important piece of both the Brumer-Stark and classic Stark conjectures as well. The fact that $-w_p\zeta'_{S,p}(0,\sigma) \in 2p\mathbb{Z}_p$ (the domain of $\exp_p$) follows from the main result of [DR]. As mentioned in the Introduction, this conjecture was recently proved in [DDP] without, however, giving a direct construction of the Gross-Stark unit α_{Gr}.

3. Computation of $\zeta_T(0,\sigma)$ and $\zeta'_{S,p}(0,\sigma)$

In order to give an effective algorithm for computing the Gross-Stark unit α_{Gr}, we first require an efficient method to compute the special values $\zeta_T(0,\sigma)$ and $\zeta'_{S,p}(0,\sigma)$. For ease of presentation, all notations and conventions used here are set up to be consistent with those used in [T] and [TY] and in Section 2 above.

In order to compute $\zeta_T(0,\sigma)$, let $\mathfrak{m}$ be an integral ideal of $\mathcal{O}_{\mathsf{F}}$ divisible to an appropriate power by every finite prime in the set T (in particular, $\overline{\mathfrak{p}} \mid \mathfrak{m}$) and let $H_+(\mathfrak{m})$ denote the narrow ray class group modulo $\mathfrak{m}$. Assume the integral ideal $\mathfrak{b} \subseteq \mathcal{O}_{\mathsf{F}}$ belongs to the fixed class $\mathcal{B}_+ \in H_+(\mathfrak{m})$. As described in [T], we may apply a continued fraction algorithm due to Zagier [Z] and Hayes [H] to produce an ordered sequence of N oriented $\mathbb{Z}$-bases $\{\gamma_0,\gamma_1\}$, $\{\gamma_1,\gamma_2\},\ldots,\{\gamma_{N-1},\gamma_N\}$ for $\mathfrak{m}\mathfrak{b}^{-1}$ such that $\mathfrak{m}\mathfrak{b}^{-1} = [\gamma_0,\gamma_1] = [\gamma_1,\gamma_2] = \cdots = [\gamma_{N-1},\gamma_N]$, $0 < \gamma_j^{(1)} < \gamma_{j-1}^{(1)}$, and $0 < \gamma_{j-1}^{(2)} < \gamma_j^{(2)}$ for all $1 \le j \le N$. Let $\beta_j = \gamma_{j-1}/\gamma_j$ and assume $1 = w_j\gamma_{j-1} + z_j\gamma_j$, $j = 1,\ldots,N$, for uniquely determined rational numbers $w_j, z_j \in \mathbb{Q}$. If $w \in \mathbb{R}$, we write $\lfloor w \rfloor$ for the floor of w, $\lceil w \rceil$ for the ceiling, and we set $\langle w \rangle = w - \lfloor w \rfloor$. We set $\{w\} = \langle w \rangle$ if $0 < \langle w \rangle < 1$ and $\{w\} = 1$ if $w \in \mathbb{Z}$. It may be shown that the set of triples $\{(\langle w_1\rangle,\langle z_1\rangle,\beta_1),\ldots,(\langle w_N\rangle,\langle z_N\rangle,\beta_N)\}$ depends only upon the class $\mathcal{B}_+$ and not upon the choice of the integral ideal $\mathfrak{b}$ within $\mathcal{B}_+$. The special value $\zeta_{\mathfrak{m}}(0,\mathcal{B}_+)$

for the partial zeta function associated to the class $\mathcal{B}_+$ at $s = 0$ may be expressed in the form

$$\zeta_{\mathfrak{m}}(0, \mathcal{B}_+) = \sum_{j=1}^{N} z_2\big(0, (\{z_j\}, \langle w_j\rangle), (\beta_j^{(1)}, \beta_j^{(2)})\big), \tag{9}$$

where $z_2\big(s, (x_1, x_2), (\omega_1, \omega_2)\big)$ is the Shintani zeta function. We have in turn

(10)

$$z_2\big(0, (\{z_j\}, \langle w_j\rangle), (\beta_j^{(1)}, \beta_j^{(2)})\big) = \frac{1}{4}\left(\frac{1}{\beta_j^{(1)}} + \frac{1}{\beta_j^{(2)}}\right) B_2(\{z_j\}) + B_1(\{z_j\}) B_1(\langle w_j\rangle) + \frac{1}{4}\big(\beta_j^{(1)} + \beta_j^{(2)}\big) B_2(\langle w_j\rangle),$$

where $B_1(x) = x - 1/2$ and $B_2(x) = x^2 - x + 1/6$ are the first two Bernoulli polynomials. We note for future reference that each individual term in Eq. (10) is a rational number.

A finite sum of the values $\zeta_{\mathfrak{m}}(0, \mathcal{B}_+)$ mentioned above gives the special value $\zeta_T(0, \sigma)$ of interest here and we may describe this sum in terms of certain ray class group characters defined on $H_+(\mathfrak{m})$. These characters are homomorphisms from $H_+(\mathfrak{m})$ to $\mathbb{C}^\times$, and we denote the set of all such homomorphisms by $\widehat{H_+(\mathfrak{m})}$. By class field theory, the abelian extension K/F corresponds uniquely to a subgroup of characters $\mathsf{X} \subseteq \widehat{H_+(\mathfrak{m})}$ with $\mathrm{Gal}(\mathsf{K}/\mathsf{F}) \cong \mathsf{X}$. A prime ideal $\mathfrak{q} \subset \mathcal{O}_\mathsf{F}$ with $(\mathfrak{q}, \mathfrak{m}) = (1)$ splits completely in K if and only if $\chi(\mathfrak{q}) = 1$ for all $\chi \in \mathsf{X}$ (a prime ideal dividing $\mathfrak{m}$ might split completely if it does not divide the conductor $\mathfrak{f}(\mathsf{K}/\mathsf{F})$ of the extension). This characterization of the primes splitting completely in a Galois extension K of F (outside of a finite number) defines K uniquely by a theorem of Bauer (see [Ja], Cor. 5.5). If $\sigma_0 \in G$ is the identity automorphism, then

$$\zeta_T(0, \sigma_0) = \sum_{\mathcal{B}_+ \in H} \zeta_{\mathfrak{m}}(0, \mathcal{B}_+), \tag{11}$$

where H is the subgroup of elements in $H_+(\mathfrak{m})$ on which each $\chi \in \mathsf{X}$ evaluates to 1. For other $\sigma \in G$, $\zeta_T(0, \sigma)$ is obtained by taking the sum over all classes in a coset of $H_+(\mathfrak{m})/H$. Combining (9), (10), and (11) gives a proof, due to Shintani [Sh1], independent of the work of Klingen [Kl] and Siegel [Si], that $\zeta_T(0, \sigma) \in \mathbb{Q}$ and also an effective algorithm for the computation of these rational numbers. An alternate method for computing these values using L-functions was given in [RT].

The continued fraction algorithm of Zagier and Hayes is used to compute $\zeta'_{S,p}(0, \sigma)$ as well, but this time we work with respect to $H_+(\mathfrak{m}\mathfrak{p})$, the narrow ray class group modulo $\mathfrak{m}\mathfrak{p}$. If the ideal $\mathfrak{c} \subseteq \mathcal{O}_\mathsf{F}$ belongs to the class $\mathcal{C}_+ \in H_+(\mathfrak{m}\mathfrak{p})$, we have an associated sequence of γ's corresponding to the fractional ideal $\mathfrak{m}\mathfrak{p}\mathfrak{c}^{-1}$ and a set of triples $\{(\langle w_1\rangle, \langle z_1\rangle, \beta_1), \ldots, (\langle w_M\rangle, \langle z_M\rangle, \beta_M)\}$ dependent only upon the class $\mathcal{C}_+$. The first complex derivative at $s = 0$ is given by (see [T], p. 304)

$$\zeta'_{\mathfrak{m}\mathfrak{p}}(0, \mathcal{C}_+) = \sum_{j=1}^{M} \Big[\big(-\log(\mathrm{N}(\mathfrak{c}(\gamma_j)))\big)\, z_2\big(0, (\{z_j\}, \langle w_j\rangle), (\beta_j^{(1)}, \beta_j^{(2)})\big) + z'_2\big(0, (\{z_j\}, \langle w_j\rangle), (\beta_j^{(1)}, \beta_j^{(2)})\big)\Big], \tag{12}$$

where $\mathrm{N}(\mathfrak{c}(\gamma_j)) \in \mathbb{Q}$ denotes the norm of the fractional ideal $\mathfrak{c}(\gamma_j)$. The first expression under the summation sign in Eq. (12) has an immediate interpretation as an element of $\mathbb{Q}_p$ when we replace log by the Iwasawa p-adic logarithm and recall that $z_2\big(0,(\{z_j\},\langle w_j\rangle),(\beta_j^{(1)},\beta_j^{(2)})\big) \in \mathbb{Q}$. In order to give an explicit evaluation of $z_2'\big(0,(\{z_j\},\langle w_j\rangle),(\beta_j^{(1)},\beta_j^{(2)})\big)$ as well as an eventual p-adic interpretation of this expression, we must first introduce the double zeta and double gamma functions of Barnes [B].

If x,ω_1, and ω_2 are positive real-valued parameters and $\Re(s) > 2$, the Dirichlet series

$$\zeta_2\big(s,x,(\omega_1,\omega_2)\big) := \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(x+m\omega_1+n\omega_2)^{-s}$$

converges absolutely to a function having a meromorphic continuation to the whole complex plane known as the Barnes double zeta function. This function has simple poles at $s=1$ and $s=2$ and we define the normalized double gamma function $\Gamma_2\left(x,(\omega_1,\omega_2)\right)$ [KK] by

$$\left\{\frac{\partial}{\partial s}\zeta_2\big(s,x,(\omega_1,\omega_2)\big)\right\}_{s=0} = \log\left(\Gamma_2\left(x,(\omega_1,\omega_2)\right)\right). \tag{13}$$

The following formula is due to Shintani (see [Sh2], p. 176):

$$\begin{aligned} z_2'\big(0,(\{z_j\},\langle w_j\rangle),(\beta_j^{(1)},\beta_j^{(2)})\big) &= \log\left(\Gamma_2\left(\{z_j\}+\langle w_j\rangle\beta_j^{(1)},\left(1,\beta_j^{(1)}\right)\right)\right) \\ &+ \log\left(\Gamma_2\left(\{z_j\}+\langle w_j\rangle\beta_j^{(2)},\left(1,\beta_j^{(2)}\right)\right)\right) \\ &+ \frac{(\beta_j^{(1)}-\beta_j^{(2)})}{4\beta_j^{(1)}\beta_j^{(2)}}\log\left(\frac{\beta_j^{(2)}}{\beta_j^{(1)}}\right)B_2(\{z_j\}). \end{aligned} \tag{14}$$

By the result of Kashio mentioned in the Introduction ([K], Theorem 6.2), the first derivative of the p-adic version $\zeta_{\mathfrak{m}\mathfrak{p},p}(s,\mathcal{C}_+)$ of $\zeta_{\mathfrak{m}\mathfrak{p}}(s,\mathcal{C}_+)$ evaluated at $s=0$ is also given by (12) and (14) combined, once all of the terms are interpreted properly in a p-adic manner. By the comment immediately following Eq. (12), only the terms on the right side of (14) remain to be p-adically interpreted. By assumption, the prime p splits completely in F with $(p)=\mathfrak{p}\overline{\mathfrak{p}}$, which implies that there are two distinct embeddings of F into $\mathbb{Q}_p$ corresponding respectively to the two distinct prime ideals $\mathfrak{p}$ and $\overline{\mathfrak{p}}$. Let $x_{\mathfrak{p}}$ and $x_{\overline{\mathfrak{p}}}$ denote the image of $x \in \mathsf{F}$ with respect to the embedding $\mathsf{F} \hookrightarrow \mathbb{Q}_p$ corresponding to $\mathfrak{p}$ and $\overline{\mathfrak{p}}$, respectively. The first step towards p-adically interpreting the terms on the right side of (14) is to replace $\beta_j^{(1)}$ everywhere by $(\beta_j)_{\mathfrak{p}}$ and $\beta_j^{(2)}$ everywhere by $(\beta_j)_{\overline{\mathfrak{p}}}$. Again, replacing log by the Iwasawa p-adic logarithm allows an immediate p-adic interpretation of the bottom expression on the right side of (14).

The problem of developing a p-adic counterpart to the function $\log\left(\Gamma_2\left(x,(\omega_1,\omega_2)\right)\right)$ was discussed at length in [TY]. The continued fraction algorithm of Zagier and Hayes leads to quantities $x,\omega_1,\omega_2 \in \mathbb{Q}_p$ in all terms on the right side of (14) that satisfy

$$|x|_p > \max\{|\omega_1|_p, |\omega_2|_p\}, \tag{15}$$

where $|x|_p$ denotes the p-adic absolute value of x normalized by $|p|_p = p^{-1}$. This is important for two reasons. First, when $x,\omega_1,\omega_2 \in \mathbb{Q}_p$ satisfy (15), then $|x|_p > 1$ since $\omega_1 = 1$ in each term on the right side of (14). In this case, it follows that the p-adic counterpart $G_{p,2}(x,(\omega_1,\omega_2))$ to $\log\left(\Gamma_2\left(x,(\omega_1,\omega_2)\right)\right)$ we defined in [TY] agrees exactly with the somewhat different p-adic counterpart $L\Gamma_{p,2}(x,(\omega_1,\omega_2))$ that Kashio ([K], p. 114) has defined (see the remark made at the beginning of Section 4 of [TY]). This allows us to apply his Theorem 6.2 ([K], p. 121) with the special values of our functions in place of his. This leads to the second reason, namely, there is a very efficient formula (see [TY], Theorem 4.2) for computing the p-adic counterpart $G_{p,2}(x,(\omega_1,\omega_2))$ when x,ω_1,ω_2 satisfy (15), thus finally giving an effective means to compute $\zeta'_{\mathfrak{mp},p}(0,\mathcal{C}_+)$. We note that the formula in Theorem 4.2 of [TY] was directly inspired by the connection between formula (5) in the Introduction to Diamond's p-adic log gamma function $G_p(\frac{x}{f})$. The following result summarizes the comments above.

Proposition 1. *With reference to Equations (12) and (14),*

$$|\{z_j\}+\langle w_j\rangle(\beta_j)_{\mathfrak{p}}|_p > \max\{1,|(\beta_j)_{\mathfrak{p}}|_p\} \quad \textit{and} \quad |\{z_j\}+\langle w_j\rangle(\beta_j)_{\overline{\mathfrak{p}}}|_p > \max\{1,|(\beta_j)_{\overline{\mathfrak{p}}}|_p\}$$

for $j=1,\ldots,M$.

Proof. We restrict ourselves to proving that $|\{z_1\}+\langle w_1\rangle(\beta_1)_{\mathfrak{p}}|_p > \max\{1,|(\beta_1)_{\mathfrak{p}}|_p\}$, the proof being the same in all cases. Since $\gamma_1\in\mathfrak{mpc}^{-1}$, there exists an integral ideal $\mathfrak{a}\subseteq\mathcal{O}_{\mathsf{F}}$ such that $\mathfrak{mpc}^{-1}\cdot\mathfrak{a}=(\gamma_1)$ or $\mathfrak{mpa}=\mathfrak{c}(\gamma_1)$. This implies that $\mathrm{ord}_{\mathfrak{p}}(\gamma_1)\geq 1$ since $\mathfrak{p}\nmid\mathfrak{c}$ and so $|(\gamma_1)_{\mathfrak{p}}|_p<1$. Similarly, $|(\gamma_0)_{\mathfrak{p}}|_p<1$. Since $1=z_1\gamma_1+w_1\gamma_0$, we have $1=\{z_1\}\gamma_1+\langle w_1\rangle\gamma_0+s\gamma_0+t\gamma_1$ with $s,t\in\mathbb{Z}$ and so $(\{z_1\}\gamma_1+\langle w_1\rangle\gamma_0)_{\mathfrak{p}}\in\mathbb{Z}_p$. We conclude that $|(\{z_1\}\gamma_1+\langle w_1\rangle\gamma_0)_{\mathfrak{p}}|_p=1>\max\{|(\gamma_0)_{\mathfrak{p}}|_p,|(\gamma_1)_{\mathfrak{p}}|_p\}$. Dividing both sides by $|(\gamma_1)_{\mathfrak{p}}|_p$ yields $|\{z_1\}+\langle w_1\rangle(\beta_1)_{\mathfrak{p}}|_p>\max\{1,|(\beta_1)_{\mathfrak{p}}|_p\}$. Since $\overline{\mathfrak{p}}\mid\mathfrak{m}$, the inequality $|\{z_1\}+\langle w_1\rangle(\beta_1)_{\overline{\mathfrak{p}}}|_p>\max\{1,|(\beta_1)_{\overline{\mathfrak{p}}}|_p\}$ also holds by the same proof. □

Define $\nu:=\mathrm{N}(\mathfrak{mp})-1$ and let $[\nu]_+$ denote the narrow class modulo $\mathfrak{mp}$ to which the principal ideal (ν) belongs. In Proposition 1 of [T], an important connection due to Shintani between the values $\zeta'_{\mathfrak{mp}}(0,\mathcal{C}_+)$ and $\zeta'_{\mathfrak{mp}}(0,[\nu]_+\mathcal{C}_+)$ for each class $\mathcal{C}_+\in H_+(\mathfrak{mp})$ was stated and we now show that an interesting relation exists in the p-adic situation as well.

Proposition 2. *For each class* $\mathcal{C}_+\in H_+(\mathfrak{mp})$ *we have*

$$\zeta'_{\mathfrak{mp},p}(0,[\nu]_+\mathcal{C}_+)=\zeta'_{\mathfrak{mp},p}(0,\mathcal{C}_+).$$

Proof. Since the formula defining $\zeta'_{\mathfrak{mp},p}(0,\mathcal{C}_+)$ is essentially the same as that defining $\zeta'_{\mathfrak{mp}}(0,\mathcal{C}_+)$, we may follow the same steps as in Proposition 1 of [T] to deduce that

$$\begin{aligned}
&\zeta'_{\mathfrak{mp},p}(0,[\nu]_+\mathcal{C}_+)-\zeta'_{\mathfrak{mp},p}(0,\mathcal{C}_+)\\
&=\sum_{j=1}^{M}\Big[G_{p,2}(1+(\beta_j)_{\mathfrak{p}}-(\{z_j\}+\langle w_j\rangle(\beta_j)_{\mathfrak{p}}),(1,(\beta_j)_{\mathfrak{p}}))\\
&-G_{p,2}(\{z_j\}+\langle w_j\rangle(\beta_j)_{\mathfrak{p}},(1,(\beta_j)_{\mathfrak{p}}))\Big]\\
&+\sum_{j=1}^{M}\Big[G_{p,2}(1+(\beta_j)_{\overline{\mathfrak{p}}}-(\{z_j\}+\langle w_j\rangle(\beta_j)_{\overline{\mathfrak{p}}}),(1,(\beta_j)_{\overline{\mathfrak{p}}}))\\
&-G_{p,2}(\{z_j\}+\langle w_j\rangle(\beta_j)_{\overline{\mathfrak{p}}},(1,(\beta_j)_{\overline{\mathfrak{p}}}))\Big].
\end{aligned}$$

By the reflection functional equation derived in Theorem 3.4 (iii) of [TY], $G_{p,2}(\omega_1+\omega_2-x,(\omega_1,\omega_2)) = G_{p,2}(x,(\omega_1,\omega_2))$ and therefore the right side of the equation above is identically equal to zero. $\square$

Once the values $\zeta'_{\mathfrak{mp},p}(0,\mathcal{C}_+)$ have been computed, a finite sum of such values gives the special value $\zeta'_{S,p}(0,\sigma)$ of interest here. The same subgroup of characters $\mathsf{X} \subseteq \widehat{H_+(\mathfrak{m})}$ used to compute the values $\zeta_T(0,\sigma)$, $\sigma \in G = \mathrm{Gal}(\mathsf{K}/\mathsf{F})$, is used again here. If $\sigma_0 \in G$ is the identity automorphism, then

$$\zeta'_{S,p}(0,\sigma_0) = \sum_{\mathcal{C}_+ \in H'} \zeta'_{\mathfrak{mp},p}(0,\mathcal{C}_+), \tag{16}$$

where H' is the subgroup of elements in $H_+(\mathfrak{mp})$ on which each $\chi \in \mathsf{X}$ evaluates to 1. For other $\sigma \in G$, $\zeta'_{S,p}(0,\sigma)$ is obtained by taking the sum over all classes in a coset of $H_+(\mathfrak{mp})/H'$.

4. An example

In this section we illustrate the ideas of the previous sections by working through the details of an explicit example. All computations were carried out using the PARI/GP [GP] software package.

Let $d_{\mathsf{F}} = 29$ and let $\theta \in \overline{\mathbb{Q}}$ be a root of $f[29] = x^2 - x - 7$. The prime 13 splits into a product of two distinct prime ideals in $\mathcal{O}_{\mathsf{F}}$. Let $\mathfrak{r}$ be the prime ideal of $\mathcal{O}_{\mathsf{F}}$ lying over 13 having Hermite normal form equal to $[13,4;0,1]$ with respect to the ordered basis $\{1,\theta\}$. The prime $p = 7$ splits completely in F as well. Let $\mathfrak{p}$ and $\overline{\mathfrak{p}}$ be the prime ideals lying over 7 with respective Hermite normal form representations $[7,6;0,1]$ and $[7,0;0,1]$. The integral ideal $\mathfrak{m}$ is chosen such that $\mathfrak{m} = \mathfrak{r}\,\overline{\mathfrak{p}}$ and so $T = \{\mathfrak{p}_\infty^{(1)}, \mathfrak{p}_\infty^{(2)}, \mathfrak{r}, \overline{\mathfrak{p}}\}$. The narrow ray class group $H_+(\mathfrak{m})$ is isomorphic to $\mathbb{I}_6 \times \mathbb{I}_2$, with the class $(1,0)$ generated by the principal ideal (15) and the class $(0,1)$ generated by the prime ideal lying over 109 having Hermite normal form $[109,23;0,1]$. The ray class group characters defined on $H_+(\mathfrak{m})$ may be enumerated in the form $\chi_{j,k}$ with j and k taken modulo 6 and 2, respectively, where $\chi_{j,k}(a,b) = \rho^{ja}(-1)^{kb}$ ($\rho = \exp(2\pi i/6)$) for $(a,b) \in \mathbb{I}_6 \times \mathbb{I}_2$. The sextic character $\chi := \chi_{1,1}$ has conductor $\mathfrak{f}(\chi) = \mathfrak{mp}_\infty^{(1)}\mathfrak{p}_\infty^{(2)}$ and by class field theory there exists an abelian extension K/F corresponding to the subgroup of characters $\mathsf{X} = \langle\chi\rangle$ with $G = \mathrm{Gal}(\mathsf{K}/\mathsf{F}) \cong \mathbb{I}_6$. By the form of the conductor $\mathfrak{f}(\chi)$, K is known to be totally complex, both $\mathfrak{r}$ and $\overline{\mathfrak{p}}$ ramify in the extension K/F, and no other primes ramify. The prime ideal $\mathfrak{p}$ lies in the ray class $(3,1)$ and thus $\chi(\mathfrak{p}) = 1$, confirming that $\mathfrak{p}$ splits completely in K/F. Our goal in this section is to compute the Gross-Stark unit α_{Gr} associated to the extension K/F and the prime $p = 7$. The first step is to recognize the coefficients of

$$f_\alpha(x) = \prod_{\sigma \in G} (x - \sigma(\alpha_{\mathsf{Gr}})) \in \mathsf{F}[x] \tag{17}$$

as elements of F while working strictly within this field and using only information from F. We will prove later that for any root $\alpha \in \overline{\mathbb{Q}}$ of $f_\alpha(x)$, an explicit generation $\mathsf{K} = \mathsf{F}(\alpha)$ is obtained.

The character χ evaluates to 1 on the subgroup $H = \{(0,0),(3,1)\}$ and we find that $\zeta_T(0,\sigma_0) = 0$ by use of (9), (10), and (11). For the automorphism $\sigma \in G$ corresponding to the coset $\{(1,0),(4,1)\}$, we compute in the same way that

$\zeta_T(0,\sigma) = -2$ and furthermore that $\zeta_T(0,\sigma^2) = \zeta_T(0,\sigma^3) = 0$, $\zeta_T(0,\sigma^4) = 2$, and $\zeta_T(0,\sigma^5) = 0$.

The narrow ray class group $H_+(\mathfrak{mp})$ is isomorphic to $\mathbb{I}_6 \times \mathbb{I}_6 \times \mathbb{I}_2$, with the principal ideal $\mathfrak{c}_1 = (15)$ generating the class $(1,0,0)$, the principal ideal $\mathfrak{c}_2 = (22+2\theta)$ generates the class $(0,1,0)$, and the integral ideal $\mathfrak{c}_3$ having Hermite normal form $[335, 316; 0, 1]$ is a generator for the class $(0,0,1)$. The character values $\chi(\mathfrak{c}_1) = \rho$, $\chi(\mathfrak{c}_2) = \chi(\mathfrak{c}_3) = 1$ uniquely pin down the subgroup H' in Eq. (16). In order to recognize the coefficients of $f_\alpha(x)$ as elements in the field F, we must compute the six Galois conjugates of α_{Gr}, given as elements of $\mathbb{Q}_p$ by the formula on the right side of the equation in part 1 of Gross's conjecture, to sufficient p-adic accuracy. This prompts the question: How much p-adic accuracy is needed to recognize the coefficients of $f_\alpha(x)$ as elements of F?

We will address this question first and then proceed later to give further details on the computation of the first derivatives $\zeta'_{\mathfrak{mp},p}(0,\mathcal{C}_+)$. Let α_{Gr} be the unique element of $\mathsf{K}^\times$ satisfying Gross's conjecture. The fact that $\alpha_{\mathsf{Gr}} \in U_{\mathfrak{p}}$ places severe restrictions upon the coefficients of $f_\alpha(x)$, as we now demonstrate. We noted earlier that this condition implies that α_{Gr} has absolute value equal to one with respect to every complex absolute value of the top field K. Fix an embedding $j_1 : \mathsf{K} \hookrightarrow \mathbb{C}$ such that $j_1(\beta) = i_1(\beta)$ (see Section 2 for the definition of i_1) for all $\beta \in \mathsf{F}$ and set $|\gamma|_1 := |j_1(\gamma)|$ for all $\gamma \in \mathsf{K}$ (the absolute value symbol $|\ |$ with no subscript is always taken to mean the usual absolute value on $\mathbb{C}$). We use similar notation for a fixed embedding $j_2 : \mathsf{K} \hookrightarrow \mathbb{C}$ extending the map $i_2 : \mathsf{F} \hookrightarrow \mathbb{R}$. As an example, we consider the trace coefficient $\sum_{\sigma\in G} \sigma(\alpha_{\mathsf{Gr}})$. Assuming the general case where $n = |G|$ and $f_\alpha(x) = x^n - \lambda_{n-1}x^{n-1} + \cdots - \lambda_1 x + \lambda_0 \in \mathsf{F}[x]$ (recall from Section 1 that $2 \mid n$), we have

$$\Big|\sum_{\sigma\in G} \sigma(\alpha_{\mathsf{Gr}})\Big|_k \le \sum_{\sigma\in G} |\sigma(\alpha_{\mathsf{Gr}})|_k = n$$

for $k = 1,2$. For $\lambda_{n-1} = \sum_{\sigma\in G} \sigma(\alpha_{\mathsf{Gr}}) = a_{n-1} + b_{n-1}\theta \in \mathsf{F}$ with $a_{n-1}, b_{n-1} \in \mathbb{Q}$, we wish to determine a_{n-1} and b_{n-1}. From above, $|a_{n-1} + b_{n-1}\theta^{(k)}| \le n$ for $k = 1,2$ and so

$$|b_{n-1}\sqrt{d_{\mathsf{F}}}| = |b_{n-1}(\theta^{(1)} - \theta^{(2)})| = |(a_{n-1} + b_{n-1}\theta^{(1)}) - (a_{n-1} + b_{n-1}\theta^{(2)})| \le 2n,$$

or $|b_{n-1}| \le 2n/\sqrt{d_{\mathsf{F}}}$. By adding, instead of subtracting as above, we find

$$|2a_{n-1}| \le 2n \quad \text{if } d_{\mathsf{F}} \equiv 0 \bmod (4) \quad \text{and} \quad |2a_{n-1} + b_{n-1}| \le 2n \quad \text{if } d_{\mathsf{F}} \equiv 1 \bmod (4).$$

The following more general result follows easily.

Lemma 1. *The following bounds hold for the coefficients $\lambda_j = a_j + b_j\theta$, $a_j, b_j \in \mathbb{Q}$, of the polynomial $f_\alpha(x) = x^n - \lambda_{n-1}x^{n-1} + \cdots - \lambda_1 x + \lambda_0 \in \mathsf{F}[x]$ of which α_{Gr} is a root:*

$$|b_j| \le 2\binom{n}{j}/\sqrt{d_{\mathsf{F}}}$$

and

$$|a_j| \le \begin{cases} \binom{n}{j}, & d_{\mathsf{F}} \equiv 0 \bmod (4), \\ \binom{n}{j}(1 + 1/\sqrt{d_{\mathsf{F}}}), & d_{\mathsf{F}} \equiv 1 \bmod (4). \end{cases}$$

The following lemma also follows from the fact that $\alpha_{\mathsf{Gr}} \in U_{\mathfrak{p}}$.

Lemma 2. *Each coefficient* λ *of* $f_\alpha(x)$ *may be written as* $a + b\theta$, $a, b \in \mathbb{Q}$, *with* a *and* b *both of the form* cp^v, *where* $c, v \in \mathbb{Z}$.

Proof. Let q be a rational prime such that $q \neq p$. Assuming $\mathfrak{q} \subset \mathcal{O}_{\mathsf{F}}$ is a prime ideal lying over q whose ramification index is $e_{\mathfrak{q}}$, we define an absolute value with respect to $\mathfrak{q}$ by $|\beta|_{\mathfrak{q}} = q^{-\mathrm{ord}_{\mathfrak{q}}(\beta)/e_{\mathfrak{q}}}$ for all $\beta \in \mathsf{F}^\times$ (this absolute value restricts to the normalized absolute value on $\mathbb{Q}$ with respect to q defined by $|q|_q = q^{-1}$). Since $\alpha_{\mathsf{Gr}} \in U_{\mathfrak{p}}$, $|\lambda_j|_{\mathfrak{q}} \leq 1$ for $j = 0, \ldots, n-1$. Set $\lambda = \lambda_j$ for an arbitrary $j \in \{0, \ldots, n-1\}$ and let ϕ denote the nontrivial automorphism of $\mathrm{Gal}(\mathsf{F}/\mathbb{Q})$. Since

$$|a + b\theta|_{\mathfrak{q}} \leq 1 \quad \text{and} \quad |\phi(a + b\theta)|_{\mathfrak{q}} \leq 1, \tag{18}$$

we find by the same method as just above Lemma 1 that $|b\omega|_{\mathfrak{q}} \leq 1$, where $\omega = \theta - \phi(\theta)$ is a root of $x^2 - d_{\mathsf{F}}$ in $\overline{\mathbb{Q}}$. If $q \nmid d_{\mathsf{F}}$, then $|\omega|_{\mathfrak{q}} = 1$ and so $|b|_q \leq 1$. If $q \mid d_{\mathsf{F}}$ and q is odd, then $|\omega|_{\mathfrak{q}} = 1/\sqrt{q}$. This implies that $|b|_q \leq \sqrt{q}$ and so $|b|_q \leq 1$ since $b \in \mathbb{Q}$. For any prime $q \neq p$, if $|b|_q \leq 1$, then $|a|_q \leq 1$ since $|a + b\theta|_{\mathfrak{q}} \leq 1$ and $\theta \in \mathcal{O}_{\mathsf{F}}$. We are finally left with $d_{\mathsf{F}} \equiv 0 \bmod (4)$ and $q = 2$. Set $d_{\mathsf{F}} = 4D$, where $D > 1$ is a square-free integer and $D \equiv 2$ or $3 \bmod (4)$. We have $|\omega|_{\mathfrak{q}} = 2^{-3/2}$ or 2^{-1} according as $D \equiv 2$ or $3 \bmod (4)$. We conclude that $|b|_2 \leq 2$ since $b \in \mathbb{Q}$. Adding with respect to the inequalities in (18) gives $|2a|_2 \leq 1$ since $\theta + \phi(\theta) = 0$ in this case and so $|a|_2 \leq 2$. From above, if $|b|_2 \leq 1$, then $|a|_2 \leq 1$. Assuming that $|b|_2 = 2$ and $|a|_2 \leq 1$ leads to a contradiction since $|\theta|_{\mathfrak{q}} = 2^{-1/2}$ or 1 according as $D \equiv 2$ or $3 \bmod (4)$. If $|a|_2 \leq 1$ and $|b|_2 \leq 1$ the proof is complete, so we are left with showing that $|a|_2 = |b|_2 = 2$ is not possible. If $|a|_2 = |b|_2 = 2$, then $g = 2a$ and $h = 2b$ satisfy $|g|_2 = |h|_2 = 1$. Under this assumption, $|(a + b\theta) \cdot \phi(a + b\theta)|_{\mathfrak{q}} = |a^2 - b^2 D|_2 = 4|g^2 - h^2 D|_2$. If $\mathbb{Z}_{(2)} \subset \mathbb{Q}$ is the valuation ring at 2, then $g^2 \equiv h^2 \equiv 1 \bmod 4\mathbb{Z}_{(2)}$ and $g^2 - h^2 D \equiv 3$ or $2 \bmod 4\mathbb{Z}_{(2)}$ according as $D \equiv 2$ or $3 \bmod (4)$. This in turn implies that $|(a+b\theta) \cdot \phi(a+b\theta)|_{\mathfrak{q}} \geq 2$, contradicting the inequalities in (18). □

Returning to the more specific example under consideration with $|G| = 6$, we claim that the coefficients of $f_\alpha(x)$ satisfy the following conditions: $\lambda_5 = \lambda_1$, $\lambda_4 = \lambda_2$, and $\lambda_0 = 1$. This also follows from α_{Gr} being an element of $U_{\mathfrak{p}}$, and to see this let τ denote the unique element in G of order 2. If $\gamma \in \mathsf{K}$, then $j_1(\tau(\gamma)) = \overline{j_1(\gamma)}$ and thus

$$j_1(\alpha_{\mathsf{Gr}} \cdot \tau(\alpha_{\mathsf{Gr}})) = j_1(\alpha_{\mathsf{Gr}}) \cdot j_1(\tau(\alpha_{\mathsf{Gr}})) = j_1(\alpha_{\mathsf{Gr}}) \cdot \overline{j_1(\alpha_{\mathsf{Gr}})} = |j_1(\alpha_{\mathsf{Gr}})|^2 = 1,$$

the last equality holding since $\alpha_{\mathsf{Gr}} \in U_{\mathfrak{p}}$. Since j_1 is an embedding of K into $\mathbb{C}$, we conclude that $\alpha_{\mathsf{Gr}} \cdot \tau(\alpha_{\mathsf{Gr}}) = 1$ or $\tau(\alpha_{\mathsf{Gr}}) = 1/\alpha_{\mathsf{Gr}}$. Since G is abelian, it follows that $\tau(s(\alpha_{\mathsf{Gr}})) = 1/s(\alpha_{\mathsf{Gr}})$ for all $s \in G$, establishing the claim above concerning the coefficients of $f_\alpha(x)$. This argument works in general to prove that $f_\alpha(x)$ is palindromic by choosing τ to be the Frobenius automorphism of the infinite prime $\mathfrak{p}_\infty^{(1)}$ (or of $\mathfrak{p}_\infty^{(2)}$) with respect to the extension K/F.

We now consider the determination of the coefficient λ_5 in detail. Gross's conjecture expresses λ_5 as an element in $\mathbb{Q}_7$ in the form

$$(\lambda_5)_{\mathfrak{p}} = \sum_{\sigma \in G} (\sigma(\alpha_{\mathsf{Gr}}))_{\mathfrak{P}} \,. \tag{19}$$

The embedding $\mathsf{K} \hookrightarrow \mathbb{Q}_7$ corresponding to $\mathfrak{P}$ restricts to the embedding $\mathsf{F} \hookrightarrow \mathbb{Q}_7$ corresponding to $\mathfrak{p}$ and only this latter embedding needs to be known explicitly in order to determine the coefficients of $f_\alpha(x)$. The polynomial $f[29]$ has two roots in

$\mathbb{Q}_7$. The embedding $\mathsf{F} \hookrightarrow \mathbb{Q}_7$ corresponding to $\mathfrak{p}$ is defined by sending θ to the root $\theta_{\mathfrak{p}} = 1+7+6\cdot 7^2+7^3+2\cdot 7^4+\dots$ (the other embedding $\mathsf{F} \hookrightarrow \mathbb{Q}_7$ corresponding to $\overline{\mathfrak{p}}$ is defined by sending θ to the root $\theta_{\overline{\mathfrak{p}}} = 6\cdot 7+5\cdot 7^3+4\cdot 7^4+\dots$). From our earlier determination of the numbers $\zeta_T(0,\sigma)$, $\sigma \in G$, Gross's conjecture predicts that $|(\lambda_5)_{\mathfrak{p}}|_7 = |a_5 + b_5\theta_{\mathfrak{p}}|_7 = 7^{12}$ and we have $|a_5 + b_5\theta_{\overline{\mathfrak{p}}}|_7 \le 1$ since $\alpha_{\mathsf{Gr}} \in U_{\mathfrak{p}}$. Taking the difference yields $|b_5(\theta_{\mathfrak{p}} - \theta_{\overline{\mathfrak{p}}})|_7 = 7^{12}$ or $|b_5|_7 = 7^{12}$ since $(\theta_{\mathfrak{p}} - \theta_{\overline{\mathfrak{p}}})$ is a root of x^2-29. Adding gives $|2a_5+b_5|_7 = 7^{12}$ and so $|a_5|_7 \le 7^{12}$. By Lemma 2, we conclude that $a_5 = c_5/7^{12}$ and $b_5 = e_5/7^{12}$ with $c_5, e_5 \in \mathbb{Z}$ and $7 \nmid e_5$. Assuming we have computed the six Galois conjugates of α_{Gr} using the expressions on the right side of the equation in part 1 of Gross's conjecture accurately to at least twelve 7-adic digits, we obtain a 7-adic integer β such that $|c_5 + e_5\theta_{\mathfrak{p}} - \beta|_7 \le 7^{-12}$. Combining this with the inequality $|c_5 + e_5\theta_{\overline{\mathfrak{p}}}|_7 \le 7^{-12}$ gives $|e_5(\theta_{\mathfrak{p}} - \theta_{\overline{\mathfrak{p}}}) - \beta|_7 \le 7^{-12}$ or $|e_5 - \beta/(\theta_{\mathfrak{p}} - \theta_{\overline{\mathfrak{p}}})|_7 \le 7^{-12}$, which shows that the integer e_5 is essentially given by the expression $\beta/\sqrt{d_{\mathsf{F}}}$, in perfect analogy to the recognition process in the classic Stark conjecture setting (see [ST], bottom of p. 258). We used the p-adic version of (12) and (14) to compute

$$\begin{aligned}\beta &= 7^{12} \cdot \sum_{\sigma \in G} 7^{\,6\zeta_T(0,\sigma)} \cdot \exp_7(-6\zeta'_{S,7}(0,\sigma)) \\ &= 1+3\cdot 7+3\cdot 7^2+7^3+4\cdot 7^4+7^5+3\cdot 7^7+3\cdot 7^8 \\ &+ 6\cdot 7^9+6\cdot 7^{10}+0\cdot 7^{11}+O(7^{12}),\end{aligned}$$

and since $\theta_{\mathfrak{p}} - \theta_{\overline{\mathfrak{p}}} \equiv 1+2\cdot 7+5\cdot 7^2+3\cdot 7^3+4\cdot 7^4+5\cdot 7^5+3\cdot 7^6+4\cdot 7^8+5\cdot 7^9+5\cdot 7^{10}+2\cdot 7^{11} \bmod (7^{12})$, the integer

$$\begin{aligned}e = 3655104881 &= 1+7+3\cdot 7^2+7^4+6\cdot 7^5+7^6+4\cdot 7^8+6\cdot 7^9+5\cdot 7^{10}+7^{11} \\ &\equiv \beta/(\theta_{\mathfrak{p}} - \theta_{\overline{\mathfrak{p}}}) \bmod (7^{12})\end{aligned}$$

is our leading candidate for e_5. Since $2\binom{6}{5}/\sqrt{29} = 2.2283\dots$, Lemma 1 limits e_5 to exactly one of four choices: $e - 2\cdot 7^{12}$, $e - 7^{12}$, e, or $e + 7^{12}$. Exactly one of these choices for e_5 should be such that $\beta - e_5\theta_{\mathfrak{p}}$ is recognizable as an integer c satisfying the bound $|c| \le \binom{6}{5}(1+1/\sqrt{29})\cdot 7^{12}$ from Lemma 1. This recognition process requires β to be computed to several extra 7-adic digits of accuracy and the 7-adic expansion of c should either end in 0's or all digits being equal to $6 = p-1$ after a certain point, up to the extra accuracy of computation (the second option implies that $c_5 = c$ is negative). In this example, we found that $e_5 = e - 7^{12} = -10186182320$ and $c_5 = -849169895$. With c_5 and e_5 in hand, we may confirm the matchup between $c_5 + e_5\theta_{\mathfrak{p}}$ and β to as many p-adic digits as computed and also verify that $|c_5 + e_5\theta_{\overline{\mathfrak{p}}}|_7 \le 7^{-12}$. It is important to note that Gross's conjecture forces c_5 and e_5 to lie within a *finite* (and surprisingly small) list of possibilities. This same comment applies to every coefficient of $f_\alpha(x)$. In a similar way, we found that $\lambda_4 = (46850752816 + 989316304\,\theta)/7^{12} = \lambda_2$ and $\lambda_3 = (-1168907600 - 18302965248\,\theta)/7^{12}$. With all coefficients of $f_\alpha(x)$ now determined, an independent check may be made that any root of $f_\alpha(x)$ generates the precise abelian extension K/F under discussion from the beginning of this Section. We will return to this important point at the end of this Section.

We have seen how Gross's conjecture dictates the p-adic accuracy required to recognize any given coefficient of $f_\alpha(x)$ as an element of F and we now give further details on how the computation of the first derivatives $\zeta'_{\mathfrak{m}\mathfrak{p},p}(0,\mathcal{C}_+)$ can be carried out to a predetermined and guaranteed accuracy; again, working strictly within the field F. Based upon the discussion in Section 3, the only terms appearing in the expression for $\zeta'_{\mathfrak{mp},p}(0, \mathcal{C}_+)$ which require closer study are those involving $G_{p,2}(x, (\omega_1, \omega_2))$ (all other terms may easily be computed to any desired degree of accuracy). Assuming $x, \omega_1, \omega_2 \in \mathbb{Q}_p$ satisfy the condition $|x|_p > \max\{|\omega_1|_p, |\omega_2|_p\}$, the expansion in Theorem 4.2 of [TY] may be written out explicitly (for ease of reading, we set $C_j(\omega_1, \omega_2) := B_{2,\,j}(0; (\omega_1, \omega_2))$ in Theorem 4.2) as

$$(20)\quad G_{p,2}(x, (\omega_1, \omega_2)) = -\frac{1}{12\omega_1\omega_2}\Big[6x^2 - 6(\omega_1 + \omega_2)x + \omega_1^2 + \omega_2^2 + 3\omega_1\omega_2\Big]\log_p x + \frac{3}{4\omega_1\omega_2}x^2 - \frac{(\omega_1 + \omega_2)}{2\omega_1\omega_2}x + \sum_{j=3}^{\infty}\frac{(-1)^j(j-3)!}{j!}C_j(\omega_1, \omega_2)x^{2-j},$$

where

$$\frac{t^2}{(e^{\omega_1 t} - 1)(e^{\omega_2 t} - 1)} = \sum_{j=0}^{\infty} C_j(\omega_1, \omega_2)\frac{t^j}{j!}.$$

For example,

$$C_3(\omega_1, \omega_2) = -\frac{1}{4}(\omega_1 + \omega_2) \quad \text{and} \quad C_4(\omega_1, \omega_2) = -\frac{(\omega_1^4 - 5\omega_1^2\omega_2^2 + \omega_2^4)}{30\omega_1\omega_2}.$$

The analogy between the right sides of (5) and (20) is quite striking, with the jth Bernoulli number B_j being replaced by $C_j(\omega_1, \omega_2)$ in (20). As noted in [TY], the right side of (20) matches exactly with the asymptotic expansion of $\log(\Gamma_2(x, (\omega_1, \omega_2)))$, derived by Barnes over 100 years ago, with the error term removed!

For a fixed pair of numbers $\omega_1, \omega_2 \in \mathbb{Q}_p^\times$, we set $\bar{\omega} = (\omega_1, \omega_2)$ and define $||\bar{\omega}||_p := \max\{|\omega_1|_p, |\omega_2|_p\}$. In the course of the proof of the following proposition, we will see directly that the infinite series on the right side of (20) converges p-adically when $|x|_p > ||\bar{\omega}||_p$.

Proposition 3. *Assume $|x|_p > ||\bar{\omega}||_p$ and set $|x|_p = p^r||\bar{\omega}||_p$, where $r \in \mathbb{Z}^+$. If the infinite series in (20) is truncated after the $j = m$ term, the approximation obtained for $G_{p,2}(x, (\omega_1, \omega_2))$ is accurate to at least k p-adic digits, where*

$$k = \begin{cases} (m-1)r - 2 - \lceil \frac{\log(m+1)}{\log p} \rceil, & p > 2\,; \\ (m-1)r - 3 - \lceil \frac{\log(m+1)}{\log p} \rceil, & p = 2\,. \end{cases}$$

In particular, since $r \geq 1$, we have

$$k \geq \begin{cases} m - 3 - \lceil \frac{\log(m+1)}{\log p} \rceil, & p > 2\,; \\ m - 4 - \lceil \frac{\log(m+1)}{\log p} \rceil, & p = 2\,. \end{cases}$$

Proof. From the generating function $t/(e^{\omega t} - 1) = \sum_{k=0}^{\infty} B_k\omega^{k-1}(t^k/k!)$, we obtain

$$C_n(\omega_1, \omega_2) = \sum_{k=0}^{n}\binom{n}{k}\omega_1^{k-1}\omega_2^{n-k-1}B_kB_{n-k}.$$

By the von Staudt-Clausen Theorem, we have $|B_n|_p \leq p$ for all n, which gives the bound

$$|C_n(\omega_1, \omega_2)|_p \leq p^2||\bar{\omega}||_p^{n-2}.$$

We conclude that the jth term of the infinite series in (20) satisfies the bound

$$\left|\frac{(-1)^j C_j(\omega_1,\omega_2)x^{2-j}}{j(j-1)(j-2)}\right|_p \le E_j := p^{2+(2-j)r}\left|\frac{1}{j(j-1)(j-2)}\right|_p .$$

If $p^{s-1} < j \le p^s$, we have $s = \lceil \log j/\log p\rceil$ and

$$\left|\frac{1}{j(j-1)(j-2)}\right|_p \le \left|\frac{1}{p^s(p^s-1)(p^s-2)}\right|_p = \begin{cases} p^s, & p>2\,;\\ p^{s+1}, & p=2\,.\end{cases}$$

Therefore, truncation at the $j = m$ term gives an error bounded in absolute value by

$$\max_{j>m} E_j \le \begin{cases} p^{2+s+(1-m)r}, & p>2\,;\\ p^{3+s+(1-m)r}, & p=2\,,\end{cases}$$

where $s = \lceil \log(m+1)/\log p\rceil$, thus giving the result. □

Returning to our relative sextic example, we computed the following special values using the method described in Section 3 in conjunction with Proposition 3:

$$\begin{aligned}\zeta'_{S,7}(0,\sigma_0) &= 6\cdot 7 + 5\cdot 7^2 + 7^3 + 6\cdot 7^4 + 5\cdot 7^5 + 3\cdot 7^6 + 5\cdot 7^9\\ &\quad + 7^{10} + 2\cdot 7^{11} + 3\cdot 7^{13} + 2\cdot 7^{15} + \dots,\\ \zeta'_{S,7}(0,\sigma) &= 3\cdot 7 + 5\cdot 7^2 + 2\cdot 7^3 + 4\cdot 7^4 + 4\cdot 7^5 + 4\cdot 7^6 + 7^7\\ &\quad + 2\cdot 7^9 + 4\cdot 7^{10} + 3\cdot 7^{11} + 7^{12} + \dots,\\ \zeta'_{S,7}(0,\sigma^2) &= 7 + 4\cdot 7^2 + 6\cdot 7^3 + 3\cdot 7^4 + 6\cdot 7^5 + 6\cdot 7^6 + 5\cdot 7^7\\ &\quad + 6\cdot 7^8 + 5\cdot 7^9 + 5\cdot 7^{11} + 4\cdot 7^{13} + \dots,\end{aligned}$$

$\zeta'_{S,7}(0,\sigma^3) = -\zeta'_{S,7}(0,\sigma_0)$, $\zeta'_{S,7}(0,\sigma^4) = -\zeta'_{S,7}(0,\sigma)$, and $\zeta'_{S,7}(0,\sigma^5) = -\zeta'_{S,7}(0,\sigma^2)$. The identities $\zeta_T(0,\tau s) = -\zeta_T(0,s)$ and $\zeta'_{S,p}(0,\tau s) = -\zeta'_{S,p}(0,s)$ for $\tau = \sigma^3$ and all $s \in G$ follow from the basic properties of partial zeta functions and are consistent with the formulation of Gross's conjecture (recall our earlier derivation that $\tau(s(\alpha_{\mathsf{Gr}})) = 1/s(\alpha_{\mathsf{Gr}})$ for all $s \in G$ in connection with proving that $f_\alpha(x)$ is palindromic). We obtained the sextic polynomial $f_\alpha(x) = x^6 + \frac{1}{7^{12}}(849169895 + 10186182320\,\theta)x^5 + \cdots + 1$ by use of these special values and we now consider the extensions of F generated by the roots of $f_\alpha(x)$. The preferred input for PARI is a polynomial with algebraic integer coefficients, so we replace $f_\alpha(x)$ by the monic polynomial $g(x) \in \mathcal{O}_{\mathsf{F}}[x]$ having $7^{12}\alpha$ as a root. We first verify (using the PARI command **nffactor**) that $g(x)$ is irreducible in $\mathsf{F}[x]$. The polynomial $g(x)$ has large and unwieldy coefficients so we use the PARI command **rnfpolredabs** to obtain a new polynomal

$$h(x) = x^6 + (-\theta)x^5 + (3-2\theta)x^4 + (7+4\theta)x^3 + (3-3\theta)x^2 + (-21-3\theta)x + (15+4\theta)$$

whose roots generate the same extensions over F as both $g(x)$ and $f_\alpha(x)$. Let $\eta \in \overline{\mathbb{Q}}$ denote a fixed root of $h(x)$ and set $\mathsf{M} = \mathsf{F}(\eta)$. The PARI command **rnfinit** confirms that both of the infinite primes of F ramify in the extension M/F and computes the relative discriminant of M/F to be $\mathfrak{r}^5\overline{\mathfrak{p}}^4$.

We wish to prove that $\mathsf{M} = \mathsf{K}$, the field K being only known until now by class field theoretic considerations. The **rnfpolredabs** command allows us to recover an element $\alpha \in \mathsf{M}$ which is a root of $f_\alpha(x)$. We obtain

$$\begin{aligned}\alpha = \frac{1}{401 \cdot 7^{12}}\Big(&(140643061344\,\theta - 362736716748)\eta^5 \\ &+ (79189065448\,\theta - 1630510021112)\eta^4 \\ &+ (1506066215401\,\theta - 2812053502076)\eta^3 \\ &+ (1180925615627\,\theta + 72334114643)\eta^2 \\ &+ (516921099266\,\theta - 10079954573962)\eta \\ &+ (-1444140760423\,\theta + 3270681319416)\Big).\end{aligned} \tag{21}$$

The minimal polynomial over $\mathbb{Q}$ of η is

$$\begin{aligned} g_\eta(x) = x^{12} - x^{11} - 3x^{10} - 13x^9 + 27x^8 + 62x^7 - 103x^6 \\ + 65x^5 - 110x^4 - 124x^3 + 666x^2 - 591x + 173.\end{aligned}$$

The polynomial $g_\eta(x)$ factors in $\mathbb{Q}_7[x]$ as a product of 6 mutually distinct linear factors and one factor of degree 6 (if $\mathsf{M} = \mathsf{K}$, we expect $g_\eta(x)$ to have exactly 6 roots in $\mathbb{Q}_7$). We fix an embedding of M into $\mathbb{Q}_7$ by sending η to the particular root

$$z = 5+7+4\cdot 7^2+7^3+4\cdot 7^4+2\cdot 7^5+2\cdot 7^6+5\cdot 7^7+3\cdot 7^8+4\cdot 7^9+3\cdot 7^{10}+2\cdot 7^{11}+\dots \tag{22}$$

of $g_\eta(x)$ in $\mathbb{Q}_7$ having the effect (when taking θ to $\theta_\mathfrak{p}$) of sending α to $7^{\,6\zeta_T(0,\sigma_0)} \cdot \exp_7(-6\zeta'_{S,7}(0,\sigma_0))$, the quantity matching $(\alpha_{\mathsf{Gr}})_\mathfrak{P}$ in Gross's conjecture. Using the PARI command **nfgaloisconj**, we are able to find 6 distinct roots of $h(x)$ in the field M and to show that the corresponding Galois group $J = \mathrm{Gal}(\mathsf{M}/\mathsf{F})$ is cyclic of order 6. We are also able to choose $\upsilon \in J$ with the property that $J = \langle \upsilon \rangle$ and such that when we replace η by the root z and θ by $\theta_\mathfrak{p}$ the element

$$\begin{aligned}\upsilon(\alpha) = \frac{1}{401 \cdot 7^{13}}\Big(&(2748935557402\,\theta - 310320251941)\eta^5 \\ &+ (-1021575710803\,\theta - 15981123569411)\eta^4 \\ &+ (1999170429494\,\theta - 42154608183261)\eta^3 \\ &+ (16970309078649\,\theta + 38434638120025)\eta^2 \\ &+ (20978017191216\,\theta - 26397703223724)\eta \\ &+ (-33374757382798\,\theta - 8109194164190)\Big)\end{aligned} \tag{23}$$

is sent to $7^{\,6\zeta_T(0,\sigma)} \cdot \exp_7(-6\zeta'_{S,7}(0,\sigma))$ (recall that $\sigma \in G$ corresponds to the coset $\{(1,0),(4,1)\}$ defined at the beginning of this Section).

Now that we know that M/F is a relative abelian extension, we may prove that $\mathsf{M} = \mathsf{K}$ by use of the conductor-discriminant formula. By class field theory, the cyclic extension M/F corresponds to a group of ray class characters generated by a single character ψ of order 6. We noted above that both infinite primes of F ramify in M/F and that the relative discriminant of M/F is $\mathfrak{r}^5\overline{\mathfrak{p}}^4$. This implies that the character ψ has conductor $\mathfrak{f}(\psi) = \mathfrak{f}_\psi \mathfrak{p}_\infty^{(1)}\mathfrak{p}_\infty^{(2)}$ ($\mathfrak{f}_\psi$ is an integral ideal of $\mathcal{O}_\mathsf{F}$) and by the conductor-discriminant formula,

$$\mathfrak{r}^5\overline{\mathfrak{p}}^4 = \prod_{j=1}^{5} \mathfrak{f}_{\psi^j}\,.$$

The character ψ^3 corresponds to a relative quadratic extension E_1/F and $\mathsf{E}_1 = \mathsf{F}(\eta + \upsilon^2(\eta) + \upsilon^4(\eta))$, where $\eta + \upsilon^2(\eta) + \upsilon^4(\eta)$ is a root of the (irreducible over F) polynomial $x^2 + (-\theta)x + (9 - 2\theta)$. The relative discriminant of E_1/F is computed to be $\mathfrak{r}$ and so $\mathfrak{f}_{\psi^3} = \mathfrak{r}$ by the conductor-discriminant formula. Corresponding to the cubic character ψ^2 is a relative cubic extension E_2/F with $\mathsf{E}_2 = \mathsf{F}(\eta + \upsilon^3(\eta))$. The element $\eta + \upsilon^3(\eta) \in \mathsf{M}$ is a root of $x^3 + (-\theta)x^2 + (-7 - 2\theta)x + (14 + 7\theta)$ and the relative discriminant of E_2/F is $\mathfrak{r}^2\overline{\mathfrak{p}}^2$, which implies that $\mathfrak{f}_{\psi^2} = \mathfrak{r}\,\overline{\mathfrak{p}}$ by the conductor-discriminant formula since $\mathfrak{f}_{\psi^2} = \mathfrak{f}_{\psi^4}$ (ψ^2 and ψ^4 are conjugate characters and therefore have the same conductors). We finally conclude from above that $\mathfrak{f}_{\psi} = \mathfrak{r}\,\overline{\mathfrak{p}}$ and thus ψ is a character on the narrow ray class group $H_+(\mathfrak{m})$ defined at the beginning of this Section. There are 6 sextic characters defined on $H_+(\mathfrak{m})$ and only two of them, namely, χ and $\overline{\chi}$, have both infinite primes in their conductors. This implies that $\langle\chi\rangle = \langle\psi\rangle$ and so $\mathsf{K} = \mathsf{M}$.

We have verified that $\mathsf{K} = \mathsf{F}(\eta) = \mathbb{Q}(\eta)$ and we may now state and prove the main result of this Section. Using the PARI command **bnfinit**, we find that $w_{\mathsf{K}} = 2$.

Theorem 1. *The element $\alpha \in \mathsf{K}$ defined in (21) is equal to the Gross-Stark unit α_{Gr} associated to the extension K/F and the prime $p = 7$.*

Proof. By [DDP], we know there exists a unique element $\alpha_{\mathsf{Gr}} \in U_{\mathfrak{p}} \subset \mathsf{K}^\times$ satisfying part 1 of Gross's conjecture with respect to the embedding of K into $\mathbb{Q}_7$ induced by sending $\eta \mapsto z$ in (22). To verify that $\alpha \in U_{\mathfrak{p}}$, we first consider the absolute values at the infinite primes. Let $j_1 : \mathsf{K} \hookrightarrow \mathbb{C}$ be a fixed embedding extending $i_1 : \mathsf{F} \hookrightarrow \mathbb{R}$. Setting $\tau = \upsilon^3 \in \mathrm{Gal}(\mathsf{K}/\mathsf{F})$, we make an algebraic check that $\tau(s(\alpha)) = 1/s(\alpha)$ for all $s \in G$. The automorphism τ fixes the field E_2, which is a totally real field, and thus τ acts as complex conjugation: $j_1(\tau(\gamma)) = \overline{j_1(\gamma)}$ for all $\gamma \in \mathsf{K}$. Therefore,

$$|j_1(\alpha)|^2 = j_1(\alpha) \cdot \overline{j_1(\alpha)} = j_1(\alpha) \cdot j_1(\tau(\alpha)) = j_1(\alpha \cdot \tau(\alpha)) = j_1(1) = 1,$$

and the absolute values lying over the embedding $i_2 : \mathsf{F} \hookrightarrow \mathbb{R}$ are handled in the same way. The PARI command **idealfactor** allows us to confirm that only prime ideals above $\mathfrak{p}$ in K appear in the factorization of the principal fractional ideal (α), proving that $\alpha \in U_{\mathfrak{p}}$.

In order to confirm that $\alpha = \alpha_{\mathsf{Gr}}$, we must prove that the automorphism υ specified above corresponds to the coset $\{(1,0),(4,1)\}$ defined at the beginning of this Section. The prime ideal $\mathfrak{q} \subset \mathcal{O}_{\mathsf{F}}$ lying over 5 having Hermite normal form $[5,1;0,1]$ lies in the ray class $(1,0)$. The Frobenius automorphism $\sigma_{\mathfrak{q}}$ must be of order 6 and is therefore either equal to υ or υ^5. Let $\mathfrak{Q}$ be the unique prime ideal in $\mathcal{O}_{\mathsf{K}}$ lying over $\mathfrak{q}$. Using **idealfactor** again, we find that $\upsilon^5(\eta) - \eta^5$ is not divisible by $\mathfrak{Q}$, proving that $\upsilon = \sigma_{\mathfrak{q}}$, as desired. With respect to the unique prime ideal $\mathfrak{P} \subset \mathcal{O}_{\mathsf{K}}$ corresponding to the embedding of K into $\mathbb{Q}_7$ induced by sending $\eta \mapsto z$ in (22) we may compute the quantities $(\upsilon^j(\alpha))_{\mathfrak{P}}$, $0 \le j \le 5$, on the left side of the equation in part 1 of Gross's conjecture. We easily verify that $\mathrm{ord}_{\mathfrak{P}}(\upsilon^j(\alpha)) = 6\zeta_T(0, \upsilon^j)$ for $0 \le j \le 5$. If $\varepsilon = \alpha/\alpha_{\mathsf{Gr}}$, then $|\varepsilon|_{\mathfrak{Q}} = 1$ for *every* place $\mathfrak{Q}$ of K, which implies that ε is a root of unity in K and thus $\varepsilon = \pm 1$. We have $\alpha_{\mathfrak{P}} = 1 + 6 \cdot 7 + \ldots$, and so $\varepsilon = 1$. □

It is worth noting that α_{Gr} is a square in K in this example and so part 2 of Gross's conjecture holds automatically.

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