Explicit Computation of Gross-Stark Units over Real Quadratic Fields

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Abstract. We present an effective and practical algorithm for computing Gross-Stark units over a real quadratic base field $F$. Our algorithm allows us to explicitly construct certain relative abelian extensions of $F$ where these units lie, using only information from the base field. These units were recently proved to always exist within the correct extension fields of $F$ by Dasgupta, Darmon, and Pollack, without directly producing them.

1. Introduction

In 1981, Benedict Gross [Gr] proposed a refined conjecture concerning the values of the first derivatives of certain $p$-adic partial zeta functions at $s = 0$ which, if correct, allows one to explicitly construct relative abelian extensions of totally real algebraic number fields in the spirit of Hilbert’s 12th problem via $p$-adic analytic functions, as opposed to complex-valued such functions. A major breakthrough concerning this conjecture was very recently published in a seminal article by Dasgupta, Darmon, and Pollack [DDP]. They obtain a conditional proof of Gross’s conjecture subject essentially only to the hypothesis that Leopoldt’s conjecture holds for the totally real base field. In this paper, we study a version of Gross’s conjecture to which the proof in [DDP] applies unconditionally, namely, the base field $F$ is real quadratic and the prime $p$ splits completely in $F$. This allows us to present an effective and practical algorithm for constructing certain abelian extensions over a real quadratic field $F$ that is guaranteed to succeed based upon the results in [DDP]. This solves a problem of long-standing algorithmic interest and provides further important information at the same time. As pointed out in [DDP] (see Remark 8 on p. 443), their approach does not construct the Gross-Stark unit at the center of Gross’s conjecture in a direct fashion. In [Gr], Gross presented a proof of his conjecture for abelian extensions over $\mathbb{Q}$ as well as an explicit formula for the unit in question in terms of certain Gauss sums which are related to special values of Morita’s $p$-adic gamma function via the Gross-Koblitz formula (see also [P] for a new approach to this circle of ideas). Although it is still unknown whether analogous algebraic formulas exist for expressing the relevant Gross-Stark units over totally real base fields other than $\mathbb{Q}$, the $p$-adic version of (12) and (14) in Section 3 may be viewed as providing an analogous analytic formula in the case of real

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quadratic base fields—the role played by the Diamond function $G_p$ in Gross’s formula (see Eq. (6) below) is played in our formula by the $p$-adic log double gamma function $G_{p,2}$ developed in [TY]. The leading goal of our algorithmic approach is to obtain directly the Gross-Stark unit itself by first constructing a polynomial with coefficients in the base field of which it is a root. With this polynomial in hand, one has the means to produce the specific abelian extension of the base field where the Gross-Stark unit lies as well as being able to provide both clues and evidence towards a potential algebraic formula expressing these units over base fields other than $\mathbb{Q}$.

The main difficulty, from an algorithmic standpoint, in obtaining explicitly the Gross-Stark units over totally real fields other than $\mathbb{Q}$ is the computation of the first derivatives of the $p$-adic partial zeta functions at $s = 0$. When the base field $\mathbb{F}$ is real quadratic, there is an especially nice way to handle this computation based upon a continued fraction algorithm due to Zagier [Z] and Hayes [H] (see Section 3). For this reason, we restrict ourselves to only giving a detailed algorithm when $\mathbb{F}$ is real quadratic, however, we wish to emphasize that similar methods to those in Section 3 may be employed over higher degree totally real base fields as well. An alternate method, using different formulas, to carry out these computations was recently presented by Kashiw and Yoshida [KY1], [KY2]. Another conjecture, building upon Gross’s original conjecture, was recently proposed by Darmon and Dasgupta [DD] and also numerically studied by Dasgupta [Ds] over a real quadratic base field $\mathbb{F}$. In their conjecture, $p$ is assumed to remain inert in $\mathbb{F}$, complementing nicely the situation considered here.

Let $\mathbb{Z}$, $\mathbb{Z}^+$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{R}^+$, $\mathbb{C}$, and $\mathbb{I}_m$ denote the set of rational integers, positive integers, rational numbers, real numbers, positive real numbers, complex numbers, and $\mathbb{Z}/m\mathbb{Z}$ for a fixed integer $m \geq 2$, respectively. If $R$ is a ring with multiplicative identity $1 \neq 0$, then $R^\times$ denotes the set of units in $R$. Let $\mathbb{Q}$ denote a fixed algebraic closure of $\mathbb{Q}$, considered abstractly as opposed to being considered as a subfield of $\mathbb{C}$. If $X$ is a finite set, then $|X|$ denotes its cardinality.

The conjecture of Gross on which we focus our attention throughout is closely related to the Brumer-Stark conjecture and the set-up for both is the same: $K/\mathbb{F}$ is a relative abelian extension of number fields with $K$ totally complex and $\mathbb{F}$ totally real. Of necessity, $2 \mid n = [K : \mathbb{F}]$. These conjectures, in common with all conjectures of Stark-type, predict a precise match-up between algebraic data on one side of the equation and analytic data on the other side. If $G = \text{Gal}(K/\mathbb{F})$, then for each automorphism $\sigma \in G$ there is a partial zeta function $\zeta_T(s, \sigma)$ ($T$ is a finite set of primes in $\mathbb{F}$ whose precise definition will be given in Section 2) whose value $\zeta_T(0, \sigma)$ at $s = 0$ is a rational number. The rationality of these values was first proved by Klingen [Kl] and Siegel [Si]. If $w_K$ denotes the number of roots of unity in $K$, a general theorem due to Barsky [Ba], Cassou-Noguès [CN], and Deligne-Ribet [DR] states that $w_K\zeta_T(0, \sigma) \in \mathbb{Z}$. These integers, $w_K\zeta_T(0, \sigma)$, $\sigma \in G$, constitute the analytic data that goes into the Brumer-Stark conjecture. The conjecture of Gross, which is a $p$-adic refinement of the Brumer-Stark conjecture, requires the rational numbers $\zeta_T(0, \sigma)$, $\sigma \in G$, as well as the first derivatives evaluated at $s = 0$, $\zeta''_T(0, \sigma)$ (the set $S$ is similar to $T$ above), of certain $p$-adic partial zeta functions, one defined for each $\sigma \in G$. The algebraic data in each case is of the same type: A well-defined algebraic number lying in the field $K$, denoted by $\alpha_B$ and $\alpha_G$, respectively. We will later see that these two numbers are closely related and we generically call
these α’s “Stark units”. In each conjecture, we will focus on a nonzero prime ideal \( p \) of the ring of integers \( \mathcal{O}_F \) of \( F \) that splits completely in the extension \( K/F \) and which lies above the prime \( p \in \mathbb{Z} \) used to define the \( p \)-adic partial zeta functions mentioned above. The α’s lie in the subset \( U_p \) of \( K \) defined by
\[
U_p = \{ \beta \in K^* : |\beta|_p = 1 \text{ if } \Omega \text{ does not divide } p \},
\]
and in particular the absolute values of the α’s with respect to every complex embedding \( K \rightarrow \mathbb{C} \) are all equal to one. The only nontrivial absolute values associated to a given α arise from the \( n = |G| \) distinct prime ideals \( \mathfrak{p} \subset \mathcal{O}_K \) lying over \( p \) (remember that \( p \) splits completely!) and these values are specified by the analytic data mentioned above: The \( n \) values \( \zeta_T(0, \sigma), \sigma \in G \).

As an introduction to the Brumer-Stark and Gross conjectures, we work through a detailed example over the base field \( \mathbb{Q} \) (for ease of presentation, all number fields in this example are considered as subfields of \( \mathbb{C} \)). The computations we carry out over real quadratic base fields follow a similar though more intricate pattern. Let \( \mathbb{L} \) denote the field of 16th roots of unity. The Galois group of the extension \( \mathbb{L}/\mathbb{Q} \) is isomorphic to \((\mathbb{Z}/16\mathbb{Z})^\times \cong \mathbb{Z}_4 \times \mathbb{Z}_2 \), which in turn is isomorphic to the ray class group modulo \((16)p_{\infty}\), where \( p_{\infty} \) denotes the unique infinite place of \( \mathbb{Q} \). Consider the following Dirichlet character \( \chi \) defined modulo 16: \( \chi(1) = 1, \chi(3) = i, \chi(5) = i, \chi(7) = 1, \chi(9) = -1, \chi(11) = -i, \chi(13) = -i, \chi(15) = -1 \), and \( \chi(2n) = 0 \) for all \( n \in \mathbb{Z} \). Note that \( \chi \) is an odd quartic character and corresponding to the cyclic group \( \langle \chi \rangle \) generated by \( \chi \) is an intermediate field \( K \) with \( \mathbb{Q} \subset K \subset \mathbb{L} \) (cf. [Wa], Chapter 3). The field \( K \) is totally complex since \( \chi \) is odd and \( G := \text{Gal}(K/\mathbb{Q}) \cong \mathbb{Z}_4 \).

We have \( K = \mathbb{Q}(\theta_{16}^K) = \mathbb{Q}(\zeta_{16} + \zeta_{16}^{-1}) = \mathbb{Q}(\theta) \), where \( \zeta_{16} = \exp(2\pi i/16) \) and \( \theta \) satisfies the irreducible polynomial \( x^4 + 4x^2 + 2 \). Let \( T \) denote the set \( \{p_{\infty}, 2\} \) of all places in \( \mathbb{Q} \) that ramify in \( K/\mathbb{Q} \) and we consider the Brumer-Stark and Gross conjectures with respect to the extension \( K/\mathbb{Q} \). A crucial ingredient, particularly in Gross’s conjecture, is the specification of a finite prime in the base field that splits completely in the relative abelian extension of interest. Since \( \chi(7) = 1 \), the prime \( p = (7) \) splits completely in \( K \) and we choose this prime as our distinguished split prime and let \( S = T \cup \{p\} \). The partial zeta function associated to the identity automorphism \( \sigma_0 \in G \) is given by
\[
\zeta_T(s, \sigma_0) = \zeta_1(s, 1, 16) + \zeta_1(s, 7, 16),
\]
where \( \zeta_1(s, x, f) = \sum_{n=0}^{\infty} (x + nf)^{-s} \) (we assume \( x, f \in \mathbb{R}^+ \) in this definition). After meromorphic continuation, we have \( \zeta_1(0, x, f) = (\frac{1}{2} - \frac{f}{x}) \), and so \( \zeta_T(0, \sigma_0) = 1/2 \).

The prime 3 is inert in \( K \) since \( \chi(3) = i \) and therefore \( G = \langle \sigma \rangle \), where \( \sigma \) is the Frobenius automorphism of 3. We have \( \zeta_T(s, \sigma) = \zeta_1(s, 3, 16) + \zeta_1(s, 5, 16) \) and \( \zeta_T(0, \sigma) = 1/2 \). Similarly, \( \zeta_T(0, \sigma^2) = \zeta_T(0, \sigma^3) = -1/2 \). For this example, \( w_K = 2 \).

If \( \mathfrak{P} \subset \mathcal{O}_K \) is a fixed prime ideal lying over \( p \), the Brumer-Stark conjecture specifies precisely the \( \mathfrak{P} \)-adic ordinal of the Galois conjugates of an element \( \alpha_{BS} \in U_p \), whose existence it predicts, as follows:
\[
(2) \quad \text{ord}_{\mathfrak{P}}(\sigma(\alpha_{BS})) = w_K \zeta_T(0, \sigma) \quad \text{for all } \sigma \in G.
\]
Note that the absolute value of \( \alpha_{BS} \) is specified at all places of \( K \) and therefore \( \alpha_{BS} \), assuming it exists, is uniquely defined up to a root of unity in \( K \) once \( \mathfrak{P} \) is fixed. The Brumer-Stark conjecture is usually stated in a more global fashion (see [RT], for example), but we have stated it here with respect to a completely split finite prime \( p \) in order to draw a more direct connection with the conjecture.
of Gross. The element $\alpha_{\mathfrak{B}_L}$ is also predicted to satisfy an additional "abelian condition" which we will return to in Section 2. The existence of $\alpha_{\mathfrak{B}_L}$ implies a relation among the ideal classes in $K$ containing the prime ideals above $p$ which is reminiscent of Stickelberger’s theorem on the factorization of Gauss sums in cyclotomic extensions of $\mathbb{Q}$. Indeed, the Brumer-Stark conjecture has been proven for totally complex abelian extensions of $\mathbb{Q}$ by use of Stickelberger’s theorem and $\alpha_{\mathfrak{B}_L}$ may be expressed in this case in terms of a normalized Gauss sum raised to a specific power (see [Ta], p. 109).

The special values $\zeta_T(0, \sigma)$, $\sigma \in G$, are computed working in the ray class group modulo $(16)p_\infty$ and to compute the values $\zeta_{S,T}(0, \sigma)$, $\sigma \in G$, we work modulo $(16 \cdot 7)p_\infty$. In order to motivate our $p$-adic computations, we first recall that as a complex-valued function we have

$$
\zeta'_{\mathfrak{I}}(0, x, f) = \left( \frac{x}{f} - \frac{1}{2} \right) \log f + \log \left\{ \frac{\Gamma \left( \frac{x}{f} \right)}{\sqrt{2\pi}} \right\},
$$

where $\Gamma(s)$ is the classical gamma function. The function $\zeta_S(s, \sigma)$ differs from $\zeta_T(s, \sigma)$ by an Euler factor, namely, $\zeta_S(s, \sigma) = (1-1/7^s) \zeta_T(s, \sigma)$, and $\zeta_S(0, \sigma)$ may be computed by adding together a finite number of expressions of the form appearing on the right side of (3). For example, with respect to the identity automorphism $\sigma_0$ we define the set $J = \{1, 17, 23, 33, 39, 55, \ldots, 103\}$ of all positive integers less than $f = 112 = 16 \cdot 7$, congruent to 1 or 7 modulo 16, and not divisible by 7, and find that

$$
\zeta_S(0, \sigma_0) = \sum_{x \in J} \log \left\{ \frac{\Gamma \left( \frac{x}{17} \right)}{\sqrt{2\pi}} \right\},
$$

(the terms involving $(\frac{x}{7} - \frac{1}{2}) \log f$ all cancel). An important result due to Kashio ([K], Theorem 6.2) says in this case that the value $\zeta_{S,T}(0, \sigma_0)$ is given by the same formula as in (4) once the "correct" $p$-adic interpretation is given to the log gamma function. The expression

$$
\left( \frac{x}{f} - \frac{1}{2} \right) \log \left( \frac{x}{f} \right) - \left( \frac{x}{7} \right) + \sum_{j=2}^{\infty} \frac{(-1)^j(j-2)!}{j!} B_j \left( \frac{x}{7} \right)^{1-j}
$$

is the asymptotic expansion (Stirling’s series) of $\log(\Gamma(x/f)/\sqrt{2\pi})$ ($B_j$ is the $j$th Bernoulli number). The infinite sum in (5) does not converge in $\mathbb{C}$ but Diamond [D] proved that it does converge $p$-adically if $|\frac{x}{f}|_p > 1$. Replacing log by the Iwasawa $p$-adic logarithm in (5), we let $G_p(\frac{x}{f})$ denote the value in the $p$-adic rationals $\mathbb{Q}_p$ equal to the expression in (5) when $\frac{x}{f} \in \mathbb{Q}$ and $|\frac{x}{f}|_p > 1$. By Kashio’s result and a straightforward computation, we obtain (compare the first equation in (6) below with (4.3) in [Gr])

$$
\zeta'_{S,T}(0, \sigma_0) = \sum_{x \in J} G_7(x/112) = 2 \cdot 7 + 4 \cdot 7^2 + 5 \cdot 7^4 + \cdots
$$

and it should be noted that (5) offers not only an elegant but also an efficient formula for computing $G_p(\frac{x}{f})$. In order to state Gross’s conjecture, we fix a prime ideal $\mathfrak{p} \subset \mathcal{O}_K$ over $p = (7)$ which in this example corresponds to an embedding $K \hookrightarrow \mathbb{Q}_p$ with $p = 7$ which we denote elementwise for all $x \in K$ by $x \mapsto x\mathfrak{p} \in \mathbb{Q}_p$. 
The conjecture of Gross states that there exists an element $\alpha_G \in U_p$ such that
\begin{equation}
(\sigma(\alpha_G))_p = p^{\nu_p - \tau(0, \sigma)} \cdot \exp_p(-w_p \zeta'(0, \sigma)) \quad \text{for all } \sigma \in G,
\end{equation}
where $w_p = 6$ is the number of roots of unity in $\mathbb{Q}_p$. Note that $\alpha_G$, assuming it exists, is uniquely defined once $\mathfrak{p}$ is fixed and is equal (see [Ta], p. 136) to $\alpha_{BS}^m$ multiplied by a root of unity in $K$, where $w_p = m \cdot w_K$ ($w_K | w_p$ by the existence of the embedding $K \hookrightarrow \mathbb{Q}_p$). This is consistent with the Brumer-Stark Eq. (2) since the $p$-adic exponential function on the right side of (7) takes its values in $1 + p\mathbb{Z}_p$.

Using Gross’s refined conjecture over a real quadratic field [Gr], Chap. VI, §4 of [Ta], or [DDP], for the more general statement of the conjecture. Let $F \subseteq \overline{\mathbb{Q}}$ be a fixed real quadratic field having discriminant $d_F > 0$ and set
\begin{equation}
f[d_F] = \begin{cases}
x^2 - d_F/4 & \text{if } d_F \equiv 0 \text{ mod } 4, \\
x^2 - x - (d_F - 1)/4 & \text{if } d_F \equiv 1 \text{ mod } 4.
\end{cases}
\end{equation}

If $\theta \in \overline{\mathbb{Q}}$ is a root of $f[d_F]$, then $F = \mathbb{Q}(\theta)$ and $\mathcal{O}_F = [1, \theta]$, where $[\alpha, \beta] := \{a\alpha + b\beta \mid a, b \in \mathbb{Z}\}$. All ideals we consider, whether fractional, integral, or prime, are always understood to be nonzero. If $a \subseteq \mathcal{O}_F$ is an integral ideal, let $Na = [\mathcal{O}_F : a]$ denote its norm. A rational prime is always assumed to be positive by default. Let $\theta^{(1)}$ denote the positive real root of $f[d_F]$ and $\theta^{(2)}$ the negative real root. The two real embeddings of $F$ into $\mathbb{R}$ are specified in the following order:

\begin{align*}
i_1 : F &\hookrightarrow \mathbb{R} \quad \text{is defined by the map } \quad a + b\theta \mapsto a + b\theta^{(1)}, \quad (a, b \in \mathbb{Q}), \\
i_2 : F &\hookrightarrow \mathbb{R} \quad \text{is defined by the map } \quad a + b\theta \mapsto a + b\theta^{(2)}.
\end{align*}

The two infinite primes corresponding to the two real embeddings of $F$ are denoted by $p^{(1)}_\infty$ and $p^{(2)}_\infty$, respectively. Let $p$ be a fixed rational prime that splits completely in $F$ with $(p) = \mathfrak{p}\mathfrak{p}$, $p \neq \mathfrak{p}$. We assume that $K$ is a totally complex algebraic number field, relatively abelian over $F$, and we further assume that $p$ splits completely in the extension $K/F$. Let $S$ be a finite set of places of $F$ containing $p^{(1)}_\infty$, $p^{(2)}_\infty$, $\mathfrak{p}$, $\mathfrak{p}$, as...
where the sum is taken over all integral ideals \( a \subseteq O_F \) not divisible by any finite prime in \( T \) and having the same Artin symbol \((K/F, a) = \sigma_a = \sigma\). The infinite sum on the right side of (8) converges only for \( \Re(s) > 1 \), but \( \zeta_T(s, \sigma) \) has a meromorphic continuation to all of \( \mathbb{C} \) with exactly one (simple) pole at \( s = 1 \). Let \( w_p \) denote the number of roots of unity in \( \mathbb{Q}_p \), namely, \( w_2 = 2 \) and \( w_p = p - 1 \) if \( p \) is odd. As mentioned in the Introduction, \( w_p \zeta_T(0, \sigma) \oid \in \mathbb{Z} \) for all \( \sigma \in G \). The partial zeta functions \( \zeta_S(s, \sigma), \sigma \in G \), are defined exactly as \( \zeta_T(s, \sigma) \) above. Since both prime ideals in \( O_F \) lying over \( p \) are in \( S \), a \( p \)-adic partial zeta function \( \zeta_{S,p}(s, \sigma) \) exists for each \( \sigma \in G \) whose values interpolate exactly those of \( \zeta_S(s, \sigma) \) at certain nonpositive integers (see [CN], Corollaire 23) and \( \zeta_{S,p}(s, \sigma) \) is \( p \)-adically differentiable at \( s = 0 \).

Let \( \mathfrak{P} \subseteq O_K \) be a fixed prime ideal lying over \( p \) and let \( x_{\mathfrak{P}} \) denote the image of \( x \in K \) with respect to the embedding \( K \hookrightarrow \mathbb{Q}_p \) corresponding to \( \mathfrak{P} \). Given the above notations and assumptions, we can now state

**GROSS’S REFINED CONJECTURE ([Gr]).** There exists a unique element \( \alpha_G \in U_p \subset K^\times \) such that

1. \( (\sigma(\alpha_G))_{\mathfrak{P}} = p^{w_p \zeta_T(0, \sigma)} \cdot \exp_p(-w_p \zeta_{S,p}^\prime(0, \sigma)) \) for all \( \sigma \in G \), and
2. \( K(1,0^p) \) is an abelian extension of \( F \).

Part 2 of this conjecture, the so-called “abelian condition”, is an important piece of both the Brumer-Stark and classic Stark conjectures as well. The fact that \(-w_p \zeta_{S,p}^\prime(0, \sigma) \in 2p\mathbb{Z}_p \) (the domain of \( \exp_p \)) follows from the main result of [DR]. As mentioned in the Introduction, this conjecture was recently proved in [DDP] without, however, giving a direct construction of the Gross-Stark unit \( \alpha_G \).

### 3. Computation of \( \zeta_T(0, \sigma) \) and \( \zeta_{S,p}^\prime(0, \sigma) \)

In order to give an efficient algorithm for computing the Gross-Stark unit \( \alpha_G \), we first require an efficient method to compute the special values \( \zeta_T(0, \sigma) \) and \( \zeta_{S,p}^\prime(0, \sigma) \). For ease of presentation, all notations and conventions used here are set up to be consistent with those used in [T] and [TY] and in Section 2 above.

In order to compute \( \zeta_T(0, \sigma) \), let \( \mathfrak{m} \) be an integral ideal of \( O_F \) divisible to an appropriate power by every finite prime in the set \( T \) (in particular, \( \mathfrak{p} \mid \mathfrak{m} \)) and let \( H_\pm(\mathfrak{m}) \) denote the narrow ray class group modulo \( \mathfrak{m} \). Assume the integral ideal \( \mathfrak{b} \subseteq O_F \) belongs to the fixed class \( \mathcal{B}_+ \subset H_+(\mathfrak{m}) \). As described in [T], we may apply a continued fraction algorithm due to Zagier [Z] and Hayes [H] to produce an ordered sequence of \( N \) oriented \( \mathbb{Z} \)-bases \( \{\gamma_0, \gamma_1\}, \{\gamma_1, \gamma_2\}, \ldots, \{\gamma_{N-1}, \gamma_N\} \) for \( \mathfrak{mb}^{-1} \) such that \( \mathfrak{mb}^{-1} = [\gamma_0, \gamma_1] = [\gamma_1, \gamma_2] = \cdots = [\gamma_{N-1}, \gamma_N] \), \( 0 < \gamma_j^{(1)} < \gamma_j^{(2)} \) for all \( 1 \leq j \leq N \). Let \( \beta_j = \gamma_j - 1/\gamma_j \) and assume \( 1 = w_j\gamma_{j-1} + z_j\gamma_j \), \( j = 1, \ldots, N \), for uniquely determined rational numbers \( w_j, z_j \in \mathbb{Q} \). If \( w \in \mathbb{R} \), we write \( [w] \) for the floor of \( w \), \( \lceil w \rceil \) for the ceiling, and we set \( \langle w \rangle = w - [w] \). We set \( \{w\} \) if \( 0 < \langle w \rangle < 1 \) and \( \{w\} = 1 \) if \( w \in \mathbb{Z} \). It may be shown that the set of triples \( \{(w_1), (z_1), (\beta_1), \ldots, (w_N), (z_N), (\beta_N)\} \) depends only upon the class \( \mathcal{B}_+ \) and not upon the choice of the integral ideal \( \mathfrak{b} \) within \( \mathcal{B}_+ \). The special value \( \zeta_m(0, \mathcal{B}_+) \)
for the partial zeta function associated to the class $B_0$ at $s = 0$ may be expressed in the form

$$
\zeta_m(0, B_0) = \sum_{j=1}^{N} z_2\left(0, \{z_j\}, \langle w_j \rangle, (\beta_j^{(1)}, \beta_j^{(2)})\right),
$$

where $z_2(s, (x_1, x_2), (\omega_1, \omega_2))$ is the Shintani zeta function. We have in turn

$$
z_2\left(0, \{\{z_j\}, \langle w_j \rangle\}, (\beta_j^{(1)}, \beta_j^{(2)})\right) = \frac{1}{4} \left( \frac{1}{\beta_j^{(1)}} + \frac{1}{\beta_j^{(2)}} \right) B_2(\{z_j\}) + B_1(\{z_j\}) B_1(\{w_j\})
$$

+ \frac{1}{4} (\beta_j^{(1)} + \beta_j^{(2)}) B_2(\{w_j\}),

where $B_1(x) = x - 1/2$ and $B_2(x) = x^2 - x + 1/6$ are the first two Bernoulli polynomials. We note for future reference that each individual term in Eq. (10) is a rational number.

A finite sum of the values $\zeta_m(0, B_+)$ mentioned above gives the special value $\zeta_T(0, \sigma)$ of interest here and we may describe this sum in terms of certain ray class group characters defined on $H_+ (m)$. These characters are homomorphisms from $H_+ (m)$ to $\mathbb{C}^\times$, and we denote the set of all such homomorphisms by $\hat{H}_+ (m)$. By class field theory, the abelian extension $K/F$ corresponds uniquely to a subgroup of characters $X \subseteq \hat{H}_+ (m)$ with $\text{Gal}(K/F) \cong X$. A prime ideal $q \subset \mathcal{O}_F$ with $(q, m) = (1)$ splits completely in $K$ if and only if $\chi(q) = 1$ for all $\chi \in X$ (a prime ideal dividing $m$ might split completely if it does not divide the conductor $f(K/F)$ of the extension). This characterization of the primes splitting completely in a Galois extension $K$ of $F$ (outside of a finite number) defines $K$ uniquely by a theorem of Bauer (see [Ja], Cor. 5.5). If $\sigma_0 \in G$ is the identity automorphism, then

$$
\zeta_T(0, \sigma_0) = \sum_{B_+ \in H} \zeta_m(0, B_+),
$$

where $H$ is the subgroup of elements in $H_+ (m)$ on which each $\chi \in X$ evaluates to 1. For other $\sigma \in G$, $\zeta_T(0, \sigma)$ is obtained by taking the sum over all classes in a coset of $H_+ (m) / H$. Combining (9), (10), and (11) gives a proof, due to Shintani [Sh1], independent of the work of Klingen [K1] and Siegel [Si], that $\zeta_T(0, \sigma) \in \mathbb{Q}$ and also an effective algorithm for the computation of these rational numbers. An alternate method for computing these values using $L$-functions was given in [RT].

The continued fraction algorithm of Zagier and Hayes is used to compute $\zeta'_{S, \rho}(0, \sigma)$ as well, but this time we work with respect to $H_+ (m \rho)$, the narrow ray class group modulo $m \rho$. If the ideal $\mathfrak{c} \subseteq \mathcal{O}_F$ belongs to the class $C_+ \in H_+ (m \rho)$, we have an associated sequence of $\gamma$’s corresponding to the fractional ideal $m \mathfrak{c}^{-1}$ and a set of triples $\{(\langle w_1 \rangle, \{z_1\}, \langle b_1 \rangle), \ldots, (\langle w_M \rangle, \{z_M\}, \langle b_M \rangle)\}$ dependent only upon the class $C_+$. The first complex derivative at $s = 0$ is given by (see [T], p. 304)

$$
\zeta'_m(0, C_+) = \sum_{j=1}^{M} \left[ (-\log(\mathcal{N}(\mathfrak{c}(\gamma_j)))) z_2\left(0, \{z_j\}, \langle w_j \rangle\right), (\beta_j^{(1)}, \beta_j^{(2)})\right] + z_2\left(0, \{z_j\}, \langle w_j \rangle\right), (\beta_j^{(1)}, \beta_j^{(2)})\right],
$$
where \( N(c(\gamma_j)) \in \mathbb{Q} \) denotes the norm of the fractional ideal \( c(\gamma_j) \). The first expression under the summation sign in Eq. (12) has an immediate interpretation as an element of \( \mathbb{Q}_p \) when we replace \( \log \) by the Iwasawa \( p \)-adic logarithm and recall that \( z_2(0, (\{z_j\}, \langle w_j \rangle), (\beta_j^{(1)}, \beta_j^{(2)})) \in \mathbb{Q} \). In order to give an explicit evaluation of \( z_2(0, (\{z_j\}, \langle w_j \rangle), (\beta_j^{(1)}, \beta_j^{(2)})) \) as well as an eventual \( p \)-adic interpretation of this expression, we must first introduce the double zeta and double gamma functions of Barnes [B].

If \( x, \omega_1, \) and \( \omega_2 \) are positive real-valued parameters and \( \Re(s) > 2 \), the Dirichlet series

\[
\zeta_2(s, x, (\omega_1, \omega_2)) := \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (x + m\omega_1 + n\omega_2)^{-s}
\]

converges absolutely to a function having a meromorphic continuation to the whole complex plane known as the Barnes double zeta function. This function has simple poles at \( s = 1 \) and \( s = 2 \) and we define the normalized double zeta function \( \Gamma_2(x, (\omega_1, \omega_2)) \) [KK] by

\[
\left\{ \frac{\partial}{\partial s} \zeta_2(s, x, (\omega_1, \omega_2)) \right\}_{s=0} = \log (\Gamma_2(x, (\omega_1, \omega_2))).
\]

The following formula is due to Shintani (see [Sh2], p. 176):

\[
z_2'(0, (\{z_j\}, \langle w_j \rangle), (\beta_j^{(1)}, \beta_j^{(2)})) = \log \left( \Gamma_2 \left( \{z_j\} + \langle w_j \rangle, \beta_j^{(1)}, 1, \beta_j^{(1)} \right) \right)
\]

\[
+ \log \left( \Gamma_2 \left( \{z_j\} + \langle w_j \rangle, \beta_j^{(2)}, 1, \beta_j^{(2)} \right) \right)
\]

\[
+ \left( \frac{\beta_j^{(1)} - \beta_j^{(2)}}{4\beta_j^{(1)}\beta_j^{(2)}} \right) \log \left( \frac{\beta_j^{(2)}}{\beta_j^{(1)}} \right) B_2(\{z_j\}).
\]

By the result of Kashio mentioned in the Introduction ([K], Theorem 6.2), the first derivative of the \( p \)-adic version \( \zeta_{mp}(s, \mathcal{C}_+) \) of \( \zeta_{mp}(s, \mathcal{C}_+) \) evaluated at \( s = 0 \) is also given by (12) and (14) combined, once all of the terms are interpreted properly in a \( p \)-adic manner. By the comment immediately following Eq. (12), only the terms on the right side of (14) remain to be \( p \)-adically interpreted. By assumption, the prime \( p \) splits completely in \( F \) with \( (p) = pF \), which implies that there are two distinct embeddings of \( F \) into \( \mathbb{Q}_p \) corresponding respectively to the two distinct prime ideals \( p \) and \( \overline{p} \). Let \( x_p \) and \( x_{\overline{p}} \) denote the image of \( x \in F \) with respect to the embedding \( F \hookrightarrow \mathbb{Q}_p \) corresponding to \( p \) and \( \overline{p} \), respectively. The first step towards \( p \)-adically interpreting the terms on the right side of (14) is to replace \( \beta_j^{(1)} \) everywhere by \( (\beta_j)_p \) and \( \beta_j^{(2)} \) everywhere by \( (\beta_j)_{\overline{p}} \). Again, replacing \( \log \) by the Iwasawa \( p \)-adic logarithm allows an immediate \( p \)-adic interpretation of the bottom expression on the right side of (14).

The problem of developing a \( p \)-adic counterpart to the function \( \log (\Gamma_2(x, (\omega_1, \omega_2))) \) was discussed at length in [TY]. The continued fraction algorithm of Zagier and Hayes leads to quantities \( x, \omega_1, \omega_2 \in \mathbb{Q}_p \) in all terms on the right side of (14) that satisfy

\[
|x|_p > \max \{|\omega_1|_p, |\omega_2|_p\},
\]

where \( |x|_p \) denotes the \( p \)-adic absolute value of \( x \) normalized by \( |p|_p = p^{-1} \). This is important for two reasons. First, when \( x, \omega_1, \omega_2 \in \mathbb{Q}_p \) satisfy (15), then \( |x|_p > 1 \) since \( \omega_1 = 1 \) in each term on the right side of (14). In this case, it follows that
the \( p \)-adic counterpart \( \Gamma_{p,2}(x, (\omega_1, \omega_2)) \) to \( \Lambda(x, (\omega_1, \omega_2)) \) we defined in [TY] agrees exactly with the somewhat different \( p \)-adic counterpart \( \Gamma_{p,2}(x, (\omega_1, \omega_2)) \) that Kashio ([K], p. 114) has defined (see the remark made at the beginning of Section 4 of [TY]). This allows us to apply his Theorem 6.2 ([K], p. 121) with the special values of our functions in place of his. This leads to the second reason, namely, there is a very efficient formula (see [TY], Theorem 4.2) for computing the \( p \)-adic \( \zeta_{mp}^\prime(0, \mathcal{C}_+) \). We note that the formula in Theorem 4.2 of [TY] was directly inspired by the connection between formula (5) in the Introduction to Diamond’s \( p \)-adic log gamma function \( \zeta_p(\frac{1}{2}) \). The following result summarizes the comments above.

**Proposition 1.** With reference to Equations (12) and (14),
\[
|\{z_j\} + \langle w_j \rangle(\beta_j) |_{p} > \max\{1, |(\beta_j) |_{p}\} \quad \text{and} \quad |\{z_j\} + \langle w_j \rangle(\beta_j) |_{\overline{p}} > \max\{1, |(\beta_j) |_{\overline{p}}\} 
\]
for \( j = 1, \ldots, M \).

**Proof.** We restrict ourselves to proving that \( |\{z_1\} + \langle w_1 \rangle(\beta_1) |_{p} > \max\{1, |(\beta_1) |_{p}\} \), the proof being the same in all cases. Since \( \gamma_1 \in \text{mpc}^{-1} \), there exists an integral ideal \( \mathfrak{a} \subseteq \mathcal{O}_F \) such that \( \text{mpc}^{-1} \cdot \mathfrak{a} = (\gamma_1) \) or \( \text{mpc} = (\gamma_1) \). This implies that \( \text{ord}_p (\gamma_1) \geq 1 \) since \( p \nmid \mathfrak{a} \) and so \( |(\gamma_1) |_{p} < 1 \). Similarly, \( |(\gamma_0) |_{p} < 1 \). Since \( 1 = z_1 \gamma_1 + w_1 \gamma_0 \), we have \( \{z_1\} + \langle w_1 \rangle \gamma_0 = \gamma_1 + t \gamma_0 \gamma_1 \) with \( s, t \in \mathbb{Z} \) and so \( |\{z_1\} + \langle w_1 \rangle \gamma_0 |_{p} \in \mathbb{Z}_p \).
We conclude that \( |\{z_1\} + \langle w_1 \rangle \gamma_0 |_{p} = 1 > \max\{1, |(\beta_1) |_{p}\} \). Dividing both sides by \( |(\gamma_1) |_{p} \), we have \( |\{z_1\} + \langle w_1 \rangle(\beta_1) |_{p} > \max\{1, |(\beta_1) |_{p}\} \). Since \( \overline{p} \mid m \), the inequality \( |\{z_1\} + \langle w_1 \rangle(\beta_1) |_{\overline{p}} > \max\{1, |(\beta_1) |_{\overline{p}}\} \) also holds by the same proof. \( \square \)

Define \( \nu := N(\text{mp}) - 1 \) and let \( [\nu]_+ \) denote the narrow class modulo \( \text{mp} \) to which the principal ideal \( (\nu) \) belongs. In Proposition 1 of [T], an important connection due to Shintani between the values \( \zeta_{mp}^\prime(0, \mathcal{C}_+) \) and \( \zeta_{mp}^\prime([\nu]_+ \mathcal{C}_+) \) for each class \( \mathcal{C}_+ \in H_+(\text{mp}) \) was stated and we now show that an interesting relation exists in the \( p \)-adic situation as well.

**Proposition 2.** For each class \( \mathcal{C}_+ \in H_+(\text{mp}) \) we have
\[
\zeta_{mp,\nu}^\prime(0, [\nu]_+ \mathcal{C}_+) = \zeta_{mp}^\prime(0, \mathcal{C}_+). 
\]

**Proof.** Since the formula defining \( \zeta_{mp,\nu}^\prime(0, \mathcal{C}_+) \) is essentially the same as that defining \( \zeta_{mp}^\prime(0, \mathcal{C}_+) \), we may follow the same steps as in Proposition 1 of [T] to deduce that
\[
\zeta_{mp,\nu}^\prime(0, [\nu]_+ \mathcal{C}_+) - \zeta_{mp}^\prime(0, \mathcal{C}_+) 
\]
\[
= \sum_{j=1}^{M} \left[ G_{p,2}(1 + \langle \beta_j \rangle |_{p} - \{z_j\} + \langle w_j \rangle(\beta_j) |_{p}, (1, (\beta_j) |_{p}) \right. 
\]
\[
- G_{p,2}(\{z_j\} + \langle w_j \rangle(\beta_j) |_{p}, (1, (\beta_j) |_{p})) 
\]
\[
+ \sum_{j=1}^{M} \left[ G_{p,2}(1 + \langle \beta_j \rangle |_{\overline{p}} - \{z_j\} + \langle w_j \rangle(\beta_j) |_{\overline{p}}, (1, (\beta_j) |_{\overline{p}}) \right] 
\]
\[
- G_{p,2}(\{z_j\} + \langle w_j \rangle(\beta_j) |_{\overline{p}}, (1, (\beta_j) |_{\overline{p}})). 
\]
By the reflection functional equation derived in Theorem 3.4 (iii) of [TY], \(G_{p,2}(\omega_1 + \omega_2 - x, (\omega_1, \omega_2)) = G_{p,2}(x, (\omega_1, \omega_2))\) and therefore the right side of the equation above is identically equal to zero. \(\square\)

Once the values \(\zeta_{mp,p}'(0, \mathcal{C}_+^0)\) have been computed, a finite sum of such values gives the special value \(\zeta'_{S,p}(0, \sigma)\) of interest here. The same subgroup of characters \(X \subseteq \hat{H}_+(\mathfrak{m})\) used to compute the values \(\zeta_T(0, \sigma), \sigma \in G = \text{Gal}(K/F)\), is used again here. If \(\sigma_0 \in G\) is the identity automorphism, then

\[
\zeta'_{S,p}(0, \sigma_0) = \sum_{\mathcal{C}_+ \in H'} \zeta_{mp,p}'(0, \mathcal{C}_+^0),
\]

where \(H'\) is the subgroup of elements in \(H_+(\mathfrak{m}p)\) on which each \(\chi \in X\) evaluates to 1. For other \(\sigma \in G\), \(\zeta'_{S,p}(0, \sigma)\) is obtained by taking the sum over all classes in a coset of \(H_+(\mathfrak{m}p)/H'\).

4. An example

In this section we illustrate the ideas of the previous sections by working through the details of an explicit example. All computations were carried out using the PARI/GP [GP] software package.

Let \(d_F = 29\) and let \(\theta \in \mathbb{Q}\) be a root of \(f[29] = x^2 - x - 7\). The prime 13 splits into a product of two distinct prime ideals in \(\mathcal{O}_F\). Let \(\mathfrak{r}\) be the prime ideal of \(\mathcal{O}_F\) lying over 13 having Hermite normal form equal to \([13, 4; 0, 1]\) with respect to the ordered basis \(\{1, \theta\}\). The prime \(p = 7\) splits completely in \(F\) as well. Let \(p\) and \(\mathfrak{p}\) be the prime ideals lying over 7 with respective Hermite normal form representations \([7, 6; 0, 1]\) and \([7, 0; 0, 1]\). The integral ideal \(\mathfrak{m}\) is chosen such that \(\mathfrak{m} = \mathfrak{r} \mathfrak{p}\) and so \(T = \{\mathfrak{p}_\infty, \mathfrak{p}_\infty^{(2)}, \mathfrak{r}, \mathfrak{p}\}\). The narrow ray class group \(H_+(\mathfrak{m})\) is isomorphic to \(\mathbb{Z}_6 \times \mathbb{Z}_2\), with the class \((1, 0)\) generated by the principal ideal \((15)\) and the class \((0, 1)\) generated by the prime ideal lying over 109 having Hermite normal form \([109, 23; 0, 1]\). The ray class group characters defined on \(H_+(\mathfrak{m})\) may be enumerated in the form \(x_{j,k}\) with \(j\) and \(k\) taken modulo 6 and 2, respectively, where \(x_{j,0}(a, b) = \rho^a(-1)^{kb}\) (\(\rho = \exp(2\pi i/6)\)) for \((a, b) \in \mathbb{Z}_6 \times \mathbb{Z}_2\). The sextic character \(\chi := \chi_{1,1}\) has conductor \(f(\chi) = mp_\infty(1)\mathfrak{p}_\infty^{(2)}\) and by class field theory there exists an abelian extension \(K/F\) corresponding to the subgroup of characters \(X = \{\chi\}\) with \(G = \text{Gal}(K/F) \cong \mathbb{Z}_6\). By the form of the conductor \(f(\chi)\), \(K\) is known to be totally complex, both \(\mathfrak{r}\) and \(\mathfrak{p}\) ramify in the extension \(K/F\), and no other primes ramify. The prime ideal \(p\) lies in the ray class \((3, 1)\) and thus \(\chi(p) = 1\), confirming that \(p\) splits completely in \(K/F\).

Our goal in this section is to compute the Gross-Stark unit \(\alpha_{Gr}\) associated to the extension \(K/F\) and the prime \(p = 7\). The first step is to recognize the coefficients of

\[
f_\alpha(x) = \prod_{\sigma \in G} (x - \sigma(\alpha_{Gr})) \in F[x]
\]
as elements of \(F\) while working strictly within this field and using only information from \(F\). We will prove later that for any root \(\alpha \in \mathbb{Q}\) of \(f_\alpha(x)\), an explicit generation \(K = F(\alpha)\) is obtained.

The character \(\chi\) evaluates to 1 on the subgroup \(H = \{(0,0), (3,1)\}\) and we find that \(\zeta_T(0, \sigma_0) = 0\) by use of (9), (10), and (11). For the automorphism \(\sigma \in G\) corresponding to the coset \{\((1,0), (4,1)\)\}, we compute in the same way that
\(\zeta_T(0, \sigma) = -2\) and furthermore that \(\zeta_T(0, \sigma^2) = \zeta_T(0, \sigma^3) = \zeta_T(0, \sigma^4) = 2\), and \(\zeta_T(0, \sigma^5) = 0\). The narrow ray class group \(H_+ (\mathfrak{m})\) is isomorphic to \(I_0 \times I_0 \times I_2\), with the principal ideal \(\mathfrak{c}_1 = (15)\) generating the class \((1, 0, 0)\), the principal ideal \(\mathfrak{c}_2 = (22 + 2\theta)\) generates the class \((0, 1, 0)\), and the integral ideal \(\mathfrak{c}_3\) having Hermite normal form \([335, 316; 0, 1]\) is a generator for the class \((0, 0, 1)\). The character values \(\chi(\mathfrak{c}_1) = \rho, \chi(\mathfrak{c}_2) = \chi(\mathfrak{c}_3) = 1\) uniquely pin down the subgroup \(H'\) in Eq. (16). In order to recognize the coefficients of \(f_\alpha(x)\) as elements in the field \(F\), we must compute the six Galois conjugates of \(\alpha_{Gr}\) given as elements of \(\mathbb{Q}_p\) by the formula on the right side of the equation in part 1 of Gross’s conjecture, to sufficient \(p\)-adic accuracy. This prompts the question: How much \(p\)-adic accuracy is needed to recognize the coefficients of \(f_\alpha(x)\) as elements of \(F\)?

We will address this question first and then proceed later to give further details on the computation of the first derivatives \(\zeta'_{mp,p}(0, \mathcal{C}+)\). Let \(\alpha_{Gr}\) be the unique element of \(K^*\) satisfying Gross’s conjecture. The fact that \(\alpha_{Gr} \in U_p\) places severe restrictions upon the coefficients of \(f_\alpha(x)\), as we now demonstrate. We noted earlier that this condition implies that \(\alpha_{Gr}\) has absolute value equal to one with respect to every complex absolute value of the top field \(K\). Fix an embedding \(j_1 : K \hookrightarrow \mathbb{C}\) such that \(j_1(\beta) = i_1(\beta)\) (see Section 2 for the definition of \(i_1\)) for all \(\beta \in F\) and set \(|\gamma|_1 := |j_1(\gamma)|\) for all \(\gamma \in K\) (the absolute value symbol \(| \cdot |\) with no subscript is always taken to mean the usual absolute value on \(\mathbb{C}\)). We use similar notation for a fixed embedding \(j_2 : K \hookrightarrow \mathbb{C}\) extending the map \(i_2 : F \hookrightarrow \mathbb{R}\). As an example, we consider the trace coefficient \(\sum_{\sigma \in G} \sigma(\alpha_{Gr})\). Assuming the general case where \(n = |G|\) and \(f_\alpha(x) = x^n - \lambda_{n-1} x^{n-1} + \cdots - \lambda_1 x + \lambda_0 \in F[x]\) (recall from Section 1 that 2 \mid n), we have

\[
\left| \sum_{\sigma \in G} \sigma(\alpha_{Gr}) \right|_k \leq \sum_{\sigma \in G} |\sigma(\alpha_{Gr})|_k = n
\]

for \(k = 1, 2\). For \(\lambda_{n-1} = \sum_{\sigma \in G} \sigma(\alpha_{Gr}) = a_{n-1} + b_{n-1} \theta \in F\) with \(a_{n-1}, b_{n-1} \in \mathbb{Q}\), we wish to determine \(a_{n-1}\) and \(b_{n-1}\). From above, \(|a_{n-1} + b_{n-1} \theta(1)| \leq n\) for \(k = 1, 2\) and so

\[|b_{n-1} \sqrt{d_F}| = |b_{n-1} (\theta^1 - \theta^2)| = |(a_{n-1} + b_{n-1} \theta(1)) - (a_{n-1} + b_{n-1} \theta(2))| \leq 2n,\]

or \(|b_{n-1}| \leq 2n / \sqrt{d_F}\). By adding, instead of subtracting as above, we find

\[2|a_{n-1}| \leq 2n \quad \text{if } d_F \equiv 0 \mod 4 \quad \text{and} \quad 2|a_{n-1} + b_{n-1}| \leq 2n \quad \text{if } d_F \equiv 1 \mod 4.\]

The following more general result follows easily.

**Lemma 1.** The following bounds hold for the coefficients \(\lambda_j = a_j + b_j \theta, a_j, b_j \in \mathbb{Q}\), of the polynomial \(f_\alpha(x) = x^n - \lambda_{n-1} x^{n-1} + \cdots - \lambda_1 x + \lambda_0 \in F[x]\) of which \(\alpha_{Gr}\) is a root:

\[|b_j| \leq 2 \binom{n}{j} / \sqrt{d_F}\]

and

\[|a_j| \leq \begin{cases} \binom{n}{j}, & d_F \equiv 0 \mod 4, \\ \binom{n}{j}(1 + 1 / \sqrt{d_F}), & d_F \equiv 1 \mod 4. \end{cases}\]

The following lemma also follows from the fact that \(\alpha_{Gr} \in U_p\).
Lemma 2. Each coefficient $\lambda$ of $f_\alpha(x)$ may be written as $a + b\theta$, $a, b \in \mathbb{Q}$, with $a$ and $b$ both of the form $cq^v$, where $c, v \in \mathbb{Z}$.

Proof. Let $q$ be a rational prime such that $q \neq p$. Assuming $q \subset \mathcal{O}_F$ is a prime ideal lying over $q$ whose ramification index is $e_q$, we define an absolute value with respect to $q$ by $|\beta|_q = q^{-\text{ord}_q(\beta)/e_q}$ for all $\beta \in F^\times$ (this absolute value restricts to the normalized absolute value on $\mathbb{Q}$ with respect to $q$ defined by $|q|_q = q^{-1}$). Since $\alpha_{Gr} \in U_p$, $|\lambda_j|_q \leq 1$ for $j = 0, \ldots, n-1$. Set $\lambda = \lambda_j$ for an arbitrary $j \in \{0, \ldots, n-1\}$ and let $\phi$ denote the nontrivial automorphism of $\text{Gal}(F/\mathbb{Q})$. Since

$$|a + b\theta|_q \leq 1 \quad \text{and} \quad |\phi(a + b\theta)|_q \leq 1,$$

we find by the same method as just above Lemma 1 that $|b\omega|_q \leq 1$, where $\omega = \theta - \phi(\theta)$ is a root of $x^2 - d_F$ in $\overline{\mathbb{Q}}$. If $q \nmid d_F$, then $|\omega|_q = 1$ and so $|b|_q \leq 1$. If $q \mid d_F$ and $q$ is odd, then $|\omega|_q = 1/\sqrt{q}$. This implies that $|b|_q \leq \sqrt{q}$ and so $|b|_q \leq 1$ since $b \in \mathbb{Q}$. For any prime $q \neq p$, if $|b|_q \leq 1$, then $|a|_q \leq 1$ since $|a + b\theta|_q \leq 1$ and $\theta \in \mathcal{O}_F$. We are finally left with $d_F \equiv 0 \text{ mod } 4$ and $q = 2$. Set $d_F = 4D$, where $D > 1$ is a square-free integer and $D \equiv 2$ or $3 \text{ mod } 4$. We have $|\omega|_q = 2^{-3/2}$ or $2^{-1}$ according as $D \equiv 2$ or $3 \text{ mod } 4$. We conclude that $|b|_q \leq 2$ since $b \in \mathbb{Q}$. Adding with respect to the inequalities in (18) gives $2|a|_2 \leq 1$ since $\theta + \phi(\theta) = 0$ in this case and so $|a|_2 \leq 2$. From above, if $|b|_2 \leq 1$, then $|a|_2 \leq 1$. Assuming that $|b|_2 = 2$ and $|a|_2 \leq 1$ leads to a contradiction since $|\theta|_q \leq 2^{-1/2}$ or $1$ according as $D \equiv 2$ or $3 \text{ mod } 4$. If $|a|_2 \leq 1$ and $|b|_2 \leq 1$ the proof is complete, so we are left with showing that $|a|_2 = |b|_2 = 2$ is not possible. If $|a|_2 = |b|_2 = 2$, then $g = 2a$ and $h = 2b$ satisfy $|g|_2 = |h|_2 = 1$. Under this assumption, $|(a + b\theta) \cdot \phi(a + b\theta)|_q = |a^2 - b^2 D|_2 = |4g^2 - h^2 D|_2$. If $Z(2) \subset \mathbb{Q}$ is the valuation ring at $2$, then $g^2 \equiv h^2 \equiv 1 \text{ mod } 4Z(2)$, and $g^2 - h^2 D \equiv 3 \text{ or } 2 \text{ mod } 4Z(2)$ according as $D \equiv 2$ or $3 \text{ mod } 4$. This in turn implies $|(a + b\theta) \cdot \phi(a + b\theta)|_q \geq 2$, contradicting the inequalities in (18). \qed

Returning to the more specific example under consideration with $|G| = 6$, we claim that the coefficients of $f_\alpha(x)$ satisfy the following conditions: $\lambda_5 = \lambda_1$, $\lambda_3 = \lambda_2$, and $\lambda_0 = 1$. This also follows from $\alpha_{Gr}$ being an element of $U_p$, and to see this let $\tau$ denote the unique element in $G$ of order 2. If $\gamma \in K$, then $j_1(\tau(\gamma)) = \bar{j}_1(\gamma)$ and thus

$$j_1(\alpha_{Gr} \cdot \tau(\alpha_{Gr})) = j_1(\alpha_{Gr}) \cdot j_1(\tau(\alpha_{Gr})) = j_1(\alpha_{Gr}) \cdot j_1(\alpha_{Gr}) = |j_1(\alpha_{Gr})|^2 = 1,$$

the last equality holding since $\alpha_{Gr} \in U_p$. Since $j_1$ is an embedding of $K$ into $\mathbb{C}$, we conclude that $\alpha_{Gr} \cdot \tau(\alpha_{Gr}) = 1$ or $\tau(\alpha_{Gr}) = 1/\alpha_{Gr}$. Since $G$ is abelian, it follows that $\tau(s(\alpha_{Gr})) = 1/s(\alpha_{Gr})$ for all $s \in G$, establishing the claim above concerning the coefficients of $f_\alpha(x)$. This argument works in general to prove that $f_\alpha(x)$ is palindromic by choosing $\tau$ to be the Frobenius automorphism of the infinite prime $p^{(1)}$ (or of $p^{(2)}$) with respect to the extension $K/F$.

We now consider the determination of the coefficient $\lambda_5$ in detail. Gross’s conjecture expresses $\lambda_5$ as an element in $\mathbb{Q}_7$ in the form

$$\lambda_5 = \sum_{s \in G} (\sigma(\alpha_{Gr})).$$

The embedding $K \hookrightarrow \mathbb{Q}_7$ corresponding to $\mathfrak{p}$ restricts to the embedding $F \hookrightarrow \mathbb{Q}_7$ corresponding to $\mathfrak{p}$ and only this latter embedding needs to be known explicitly in order to determine the coefficients of $f_\alpha(x)$. The polynomial $f[29]$ has two roots in
The embedding $F \hookrightarrow \mathbb{Q}_7$ corresponding to $p$ is defined by sending $\theta$ to the root $\theta_p = 1 + 7 + 6 \cdot 7^2 + 7^3 + 2 \cdot 7^4 + \ldots$ (the other embedding $F \hookrightarrow \mathbb{Q}_7$ corresponding to $\overline{p}$ is defined by sending $\theta$ to the root $\theta_{\overline{p}} = 6 \cdot 7 + 5 \cdot 7^3 + 4 \cdot 7^4 + \ldots$). From our earlier determination of the numbers $\zeta_7(0, \sigma), \sigma \in G$, Gross’s conjecture predicts that $|\langle \lambda_5 \rangle_p| = |a_5 + b_5 \theta_{\overline{p}}|_7 = 7^{12}$ and we have $|a_5 + b_5 \theta_{\overline{p}}|_7 \leq 1$ since $\alpha_{\mathcal{G}} \in U_p$. Taking the difference yields $|b_5(\theta_p - \theta_{\overline{p}})|_7 = 7^{12}$ or $|b_5|_7 = 7^{12}$ since $(\theta_p - \theta_{\overline{p}})$ is a root of $x^2 - 29$. Adding gives $|2a_5 + b_5|_7 = 7^{12}$ and so $|a_5|_7 \leq 7^{12}$. By Lemma 2, we conclude that $a_5 = c_5/7^{12}$ and $b_5 = c_5/7^{12}$ with $c_5, e_5 \in \mathbb{Z}$ and $7 \not| e_5$. Assuming we have computed the six Galois conjugates of $\alpha_{\mathcal{G}}$ using the expressions on the right side of the equation in part 1 of Gross’s conjecture accurately to at least twelve 7-adic digits, we obtain a 7-adic integer $\beta$ such that $|c_5 + e_5 \theta_p - \beta|_7 \leq 7^{-12}$. Combining this with the inequality $|c_5 + e_5 \theta_p|_7 \leq 7^{-12}$ gives $|c_5(\theta_p - \theta_{\overline{p}}) - \beta|_7 \leq 7^{-12}$ or $|e_5 - \beta/(\theta_p - \theta_{\overline{p}})|_7 \leq 7^{-12}$, which shows that the integer $e_5$ is essentially given by the expression $\beta/\sqrt{29}$, in perfect analogy to the recognition process in the classic Stark conjecture setting (see [ST], bottom of p. 258). We used the $p$-adic version of (12) and (14) to compute

$$β = 7^{12} \cdot \sum_{σ ∈ G} 7^{6k_ρ(0, σ)} \cdot \exp_7(-6ζ_{5,7}(0, σ))$$

$$= 1 + 3 \cdot 7 + 3 \cdot 7^2 + 7^3 + 4 \cdot 7^4 + 7^5 + 3 \cdot 7^6 + 3 \cdot 7^8 + 6 \cdot 7^9 + 6 \cdot 7^{10} + 0 \cdot 7^{11} + O(7^{12}),$$

and since $\theta_p - \theta_{\overline{p}} ≡ 1 + 2 \cdot 7 + 5 \cdot 7^2 + 3 \cdot 7^3 + 4 \cdot 7^4 + 5 \cdot 7^5 + 3 \cdot 7^6 + 4 \cdot 7^8 + 5 \cdot 7^9 + 5 \cdot 7^{10} + 2 \cdot 7^{11} \bmod (7^{12})$

$e = 3655104881 = 1 + 7 + 3 \cdot 7^2 + 7^3 + 6 \cdot 7^5 + 7^6 + 4 \cdot 7^8 + 6 \cdot 7^9 + 5 \cdot 7^{10} + 7^{11}$

$≡ β/(\theta_p - \theta_{\overline{p}}) \bmod (7^{12})$

is our leading candidate for $e_5$. Since $2\sqrt{29} = 2.2283\ldots$, Lemma 1 limits $e_5$ to exactly one of four choices: $e - 2 \cdot 7^{12}, e - 7^{12}, e, e + 7^{12}$. Exactly one of these choices for $e_5$ should be such that $β - e_5 \theta_p$ is recognizable as an integer $e$ satisfying the bound $|e| ≤ (\frac{1}{3})(1 + 1/√29) \cdot 7^{12}$ from Lemma 1. This recognition process requires $β$ to be computed to several extra 7-adic digits of accuracy and the 7-adic expansion of $e$ should either end in 0’s or all digits being equal to 6 = $p - 1$ after a certain point, up to the extra accuracy of computation (the second option implies that $c_5 = c$ is negative). In this example, we found that $e_5 = e - 7^{12} = -10186182320$ and $c_5 = -849169895$. With $c_5$ and $e_5$ in hand, we may confirm the matchup between $c_5 + e_5 \theta_p$ and $β$ to as many $p$-adic digits as computed and also verify that $|c_5 + e_5 \theta_p|_7 ≤ 7^{-12}$. It is important to note that Gross’s conjecture forces $c_5$ and $e_5$ to lie within a finite (and surprisingly small) list of possibilities. This same comment applies to every coefficient of $f_{0}(x)$. In a similar way, we found that $λ_1 = (46850752816 + 989316304 \cdot θ)/7^{12} = λ_2$ and $λ_3 = (-1168907600 + 18302965248 \cdot θ)/7^{12}$. With all coefficients of $f_{0}(x)$ now determined, an independent check may be made that any root of $f_{0}(x)$ generates the precise abelian extension $K/F$ under discussion from the beginning of this Section. We will return to this important point at the end of this Section.

We have seen how Gross’s conjecture dictates the $p$-adic accuracy required to recognize any given coefficient of $f_{0}(x)$ as an element of $F$ and we now give further details on how the computation of the first derivatives $\zeta_\mathcal{M,p,σ}(0, C_\mathcal{+})$ can be carried out to a predetermined and guaranteed accuracy; again, working strictly within
Proposition 3. Let $\omega_1, \omega_2 \in \mathbb{Q}_p$ satisfy $|x|_p > \max\{|\omega_1|_p, |\omega_2|_p\}$, the expansion in Theorem 4.2 of [TY] may be written out explicitly (for ease of reading, we set $C_j(\omega_1, \omega_2) := B_{2j}(0; (\omega_1, \omega_2))$ in Theorem 4.2) as

$$G_{p,2}(x, (\omega_1, \omega_2)) = -\frac{1}{12\omega_1 \omega_2} \left[ 6x^2 - 6(\omega_1 + \omega_2)x + \omega_1^2 + \omega_2^2 + 3\omega_1 \omega_2 \right] \log_p x$$

$$+ \frac{3}{4\omega_1 \omega_2} x^2 - \frac{2}{2\omega_1 \omega_2} x$$

$$+ \sum_{j=3}^{\infty} \frac{(-1)^j (j-3)!}{j!} C_j(\omega_1, \omega_2) x^{2-j},$$

where

$$\frac{t^2}{(e^{\omega_1 t} - 1)(e^{\omega_2 t} - 1)} = \sum_{j=0}^{\infty} C_j(\omega_1, \omega_2) \frac{t^j}{j!}.$$

For example,

$$C_3(\omega_1, \omega_2) = -\frac{1}{4} (\omega_1 + \omega_2) \quad \text{and} \quad C_4(\omega_1, \omega_2) = -\frac{(\omega_1^4 - 5\omega_1^2 \omega_2^2 + \omega_2^4)}{30\omega_1 \omega_2}.$$

The analogy between the right sides of (5) and (20) is quite striking, with the $j$th Bernoulli number $B_j$ being replaced by $C_j(\omega_1, \omega_2)$ in (20). As noted in [TY], the right side of (20) matches exactly with the asymptotic expansion of $\log (\Gamma_2 (x, (\omega_1, \omega_2)))$, derived by Barnes over 100 years ago, with the error term removed!

For a fixed pair of numbers $\omega_1, \omega_2 \in \mathbb{Q}_p$, we set $\omega = (\omega_1, \omega_2)$ and define $||\omega||_p := \max\{|\omega_1|_p, |\omega_2||_p\}$. In the course of the proof of the following proposition, we will see directly that the infinite series on the right side of (20) converges $p$-adically when $|x|_p > ||\omega||_p$.

**Proposition 3.** Assume $|x|_p > ||\omega||_p$ and set $|x|_p = p^r ||\omega||_p$, where $r \in \mathbb{Z}^+$. If the infinite series in (20) is truncated after the $j = m$ term, the approximation obtained for $G_{p,2}(x, (\omega_1, \omega_2))$ is accurate to at least $k$ $p$-adic digits, where

$$k = \begin{cases} (m-1)r - 2 - \left\lceil \frac{\log (m+1)}{\log p} \right\rceil, & p > 2; \\ (m-1)r - 3 - \left\lceil \frac{\log (m+1)}{\log p} \right\rceil, & p = 2. \end{cases}$$

In particular, since $r \geq 1$, we have

$$k \geq \begin{cases} m - 3 - \left\lceil \frac{\log (m+1)}{\log p} \right\rceil, & p > 2; \\ m - 4 - \left\lceil \frac{\log (m+1)}{\log p} \right\rceil, & p = 2. \end{cases}$$

**Proof.** From the generating function $t/(e^{\omega_1 t} - 1) = \sum_{k=0}^{\infty} B_{k,\omega} (t^{k-1} / k!)$, we obtain

$$C_n(\omega_1, \omega_2) = \sum_{k=0}^{n} \binom{n}{k} \omega_1^{k-1} \omega_2^{n-k-1} B_k B_{n-k}.$$

By the von Staudt-Clausen Theorem, we have $|B_n|_p \leq p$ for all $n$, which gives the bound

$$|C_n(\omega_1, \omega_2)|_p \leq p^2 ||\omega||_p^{n-2}.$$
We conclude that the $j$th term of the infinite series in (20) satisfies the bound
\[
\left| \frac{(-1)^jC_j(\omega_1, \omega_2)x^{2-j}}{j(j-1)(j-2)} \right|_p \leq E_j := \frac{1}{p^j} = \frac{1}{p^{2+2-j}} \leq \frac{1}{j(j-1)(j-2)}.
\]
If $p^{s-1} < j \leq p^s$, we have $s = \lfloor \log j / \log p \rfloor$ and
\[
\frac{1}{j(j-1)(j-2)} \leq \frac{1}{p^s(p^s-1)(p^s-2)} = \begin{cases} p^s, & p > 2; \\ p^{s+1}, & p = 2. \end{cases}
\]
Therefore, truncation at the $j = m$ term gives an error bounded in absolute value by
\[
\max_{j > m} E_j \leq \begin{cases} p^{2+2+(1-m)r}, & p > 2; \\ p^{3+2+(1-m)r}, & p = 2, \end{cases}
\]
where $s = \lfloor \log(m+1)/\log p \rfloor$, thus giving the result. \qed

Returning to our relative sextic example, we computed the following special values using the method described in Section 3 in conjunction with Proposition 3:
\[
\zeta_{S,\tau}(0, \sigma_0) = 6 \cdot 7^2 + 7^3 + 6 \cdot 7^4 + 5 \cdot 7^5 + 3 \cdot 7^6 + 5 \cdot 7^9 + 7^{10} + 2 \cdot 7^{11} + 3 \cdot 7^{13} + 2 \cdot 7^{15} + \ldots,
\]
\[
\zeta_{S,\tau}(0, \sigma) = 3 \cdot 7^2 + 2 \cdot 7^3 + 4 \cdot 7^4 + 4 \cdot 7^5 + 4 \cdot 7^6 + 7^7 + 2 \cdot 7^9 + 4 \cdot 7^{10} + 3 \cdot 7^{11} + 7^{12} + \ldots,
\]
\[
\zeta_{S,\tau}(0, \sigma^2) = 7 + 4 \cdot 7^2 + 6 \cdot 7^3 + 3 \cdot 7^4 + 6 \cdot 7^5 + 6 \cdot 7^6 + 5 \cdot 7^7 + 6 \cdot 7^8 + 5 \cdot 7^{11} + 4 \cdot 7^{13} + \ldots,
\]
\[
\zeta_{S,\tau}(\sigma^3) = -\zeta_{S,\tau}(0, \sigma_0), \quad \zeta_{S,\tau}(0, \sigma^4) = -\zeta_{S,\tau}(0, \sigma), \quad \text{and} \quad \zeta_{S,\tau}(0, \sigma^5) = -\zeta_{S,\tau}(0, \sigma^2).
\]
The identities $\zeta_{\tau}(0, rs) = -\zeta_{\tau}(0, s)$ and $\zeta_{\tau}(0, rs) = -\zeta_{\tau}(0, s)$ for $\tau = \sigma^3$ and all $s \in G$ follow from the basic properties of partial zeta functions and are consistent with the formulation of Gross’s conjecture (recall our earlier derivation that $\tau(s(\alpha_G)) = 1/s(\alpha_G)$ for all $s \in G$ in connection with proving that $f_\alpha(x)$ is palindromic). We obtained the sextic polynomial $f_\alpha(x) = x^6 + \frac{1}{7}(849169895 + 10186182320 \theta)x^5 + \cdots + 1$ by use of these special values and we now consider the extensions of $F$ generated by the roots of $f_\alpha(x)$. The preferred input for PARI is a polynomial with algebraic integer coefficients, so we replace $f_\alpha(x)$ by the monic polynomial $g(x) \in \mathcal{O}_F[x]$ having $\tau^{12} \alpha$ as a root. We first verify (using the PARI command `nfinit`) that $g(x)$ is irreducible in $F[x]$. The polynomial $g(x)$ has large and unwieldy coefficients so we use the PARI command `rnfpolredabs` to obtain a new polynomial
\[
h(x) = x^6 + (-\theta)x^5 + (3 - 2\theta)x^4 + (7 + 4\theta)x^3 + (3 - 3\theta)x^2 + (-21 - 3\theta)x + (15 + 4\theta)
\]
whose roots generate the same extensions over $F$ as both $g(x)$ and $f_\alpha(x)$. Let $\eta \in \overline{\mathbb{Q}}$ denote a fixed root of $h(x)$ and set $M = F(\eta)$. The PARI command `rnfinit` confirms that both of the infinite primes of $F$ ramify in the extension $M/F$ and computes the relative discriminant of $M/F$ to be $\zeta^3 \zeta_\tau^3$. We wish to prove that $M = K$, the field $K$ being only known until now by class field theoretic considerations. The `rnfpolredabs` command allows us to recover an
element $\alpha \in M$ which is a root of $f_\alpha(x)$. We obtain

$$\alpha = \frac{1}{401 \cdot 7^{12}} \left( (140643061344 \theta - 362736716748)\eta^7 \
+ (79189065448 \theta - 1630510021112)\eta^4 \
+ (1506066215401 \theta - 2812053502076)\eta^3 \
+ (1180925615627 \theta + 72334114643)\eta^2 \
+ (51692199266 \theta - 10079954573962)\eta \
+ (-1441410760423 \theta + 3270861319416) \right).$$

The minimal polynomial over $\mathbb{Q}$ of $\eta$ is

$$g_\eta(x) = x^{12} - x^{11} - 3x^{10} + 27x^8 + 62x^7 - 103x^6$$
$$+ 65x^5 - 110x^4 - 124x^3 + 666x^2 - 591x + 173.$$  

The polynomial $g_\eta(x)$ factors in $\mathbb{Q}_7[x]$ as a product of 6 mutually distinct linear factors and one factor of degree 6 (if $M = K$, we expect $g_\eta(x)$ to have exactly 6 roots in $\mathbb{Q}_7$). We fix an embedding of $M$ into $\mathbb{Q}_7$ by sending $\eta$ to the particular root $\eta \in \mathbb{Q}_7$ having the effect (when taking $\theta$ to $\theta_p$) of sending $\alpha$ to $7^6\zeta_{7^6}(\mathbb{Q}_7, \sigma_0) \cdot \exp_{7^6}(\mathbb{Q}_7, \sigma_0))$, the quantity matching $(\alpha \mathfrak{q})_{\mathfrak{Q}}$ in Gross’s conjecture. Using the PARI command `ngaloisconj`, we are able to find 6 distinct roots of $h(x)$ in the field $M$ and to show that the corresponding Galois group $J = \text{Gal}(M/F)$ is cyclic of order 6. We are also able to choose $v \in J$ with the property that $J = \langle v \rangle$ and such that when we replace $\eta$ by the root $z$ and $\theta$ by $\theta_p$ the element

$$v(\alpha) = \frac{1}{401 \cdot 7^{13}} \left( (2748935557402 \theta - 310320251941)\eta^5 \
+ (-1021575710803 \theta - 15981123569411)\eta^4 \
+ (1999170429494 \theta - 42154608183261)\eta^3 \
+ (16970309078649 \theta + 38434638120025)\eta^2 \
+ (20978017191216 \theta - 26397703223724)\eta \
+ (-33374575382798 \theta - 8109194164190) \right)$$

is sent to $7^6\zeta_{7^6}(\mathbb{Q}_7, \sigma_0) \cdot \exp_{7^6}(\mathbb{Q}_7, \sigma_0))$ (recall that $\sigma \in G$ corresponds to the coset $\{(1,0), (4,1)\}$ defined at the beginning of this Section).

Now that we know that $M/F$ is a relative abelian extension, we may prove that $M = K$ by use of the conductor-discriminant formula. By class field theory, the cyclic extension $M/F$ corresponds to a group of ray class characters generated by a single character $\psi$ of order 6. We noted above that both infinite primes of $F$ ramify in $M/F$ and that the relative discriminant of $M/F$ is $r^5p^4$. This implies that the character $\psi$ has conductor $f(\psi) = f_\psi p_\infty^{(1)} p_\infty^{(2)}$ ($f_\psi$ is an integral ideal of $\mathcal{O}_F$) and by the conductor-discriminant formula,

$$r^5p^4 = \prod_{j=1}^{5} f_\psi.$$
The character \( \psi^3 \) corresponds to a relative quadratic extension \( E_2/F \) and \( E_1 = F(\eta + \psi^3(\eta)) \), where \( \eta + \psi^2(\eta) + \psi^4(\eta) \) is a root of the (irreducible over \( F \)) polynomial \( x^2 + (\theta) x + (9 - 2\theta) \). The relative discriminant of \( E_1/F \) is computed to be \( \tau \) and so \( f_{\psi^3} = \tau \) by the conductor-discriminant formula. Corresponding to the cubic character \( \psi^2 \) is a relative cubic extension \( E_2/F \) with \( E_2 = F(\eta + \psi^3(\eta)) \). The element \( \eta + \psi^3(\eta) \in M \) is a root of \( x^3 + (\theta) x^2 + (-7 - 2\theta) x + (14 + 7\theta) \) and the relative discriminant of \( E_2/F \) is \( \tau^2 \overline{p}^2 \), which implies that \( f_{\psi^2} = \tau \overline{p} \) by the conductor-discriminant formula since \( f_{\psi^2} = f_{\psi^4} \) (\( \psi^2 \) and \( \psi^4 \) are conjugate characters and therefore have the same conductors). We finally conclude from above that \( f_\psi = \tau \overline{p} \) and thus \( \psi \) is a character on the narrow ray class group \( H_+(m) \) defined at the beginning of this Section. There are 6 sextic characters defined on \( H_+(m) \) and only two of them, namely, \( \chi \) and \( \overline{\chi} \), have both infinite primes in their conductors. This implies that \( \langle \chi \rangle = \langle \overline{\chi} \rangle \) and so \( K = M \).

We have verified that \( K = F(\eta) = \mathbb{Q}(\eta) \) and we may now state and prove the main result of this Section. Using the PARI command \texttt{bnfinit}, we find that \( \nu_K = 2 \).

**Theorem 1.** The element \( \alpha \in K \) defined in (21) is equal to the Gross-Stark unit \( \alpha_{Gr} \) associated to the extension \( K/F \) and the prime \( p = 7 \).

**Proof.** By [DDP], we know there exists a unique element \( \alpha_{Gr} \in U_p \subset K^* \) satisfying part 1 of Gross’s conjecture with respect to the embedding of \( K \) into \( \mathbb{Q}_7 \) induced by sending \( \eta \mapsto z \) in (22). To verify that \( \alpha \in U_p \), we first consider the absolute values at the infinite primes. Let \( j_1 : K \hookrightarrow \mathbb{C} \) be a fixed embedding extending \( t_1 : F \hookrightarrow \mathbb{R} \). Setting \( \tau = v^3 \in \text{Gal}(K/F) \), we make an algebraic check that \( \tau(s(\alpha)) = 1/s(\alpha) \) for all \( s \in G \). The automorphism \( \tau \) fixes the field \( E_2 \), which is a totally real field, and thus \( \tau \) acts as complex conjugation: \( j_1(\tau(\gamma)) = j_1(\overline{\gamma}) \) for all \( \gamma \in K \). Therefore,

\[
|j_1(\alpha)|^2 = j_1(\alpha) \cdot j_1(\overline{\alpha}) = j_1(\alpha) \cdot j_1(\tau(\alpha)) = j_1(\alpha \cdot \tau(\alpha)) = j_1(1) = 1,
\]

and the absolute values lying over the embedding \( i_2 : F \hookrightarrow \mathbb{R} \) are handled in the same way. The PARI command \texttt{idealfactor} allows us to confirm that only prime ideals above \( p \) in \( K \) appear in the factorization of the principal fractional ideal \( \langle \alpha \rangle \), proving that \( \alpha \in U_p \).

In order to confirm that \( \alpha = \alpha_{Gr} \), we must prove that the automorphism \( \nu \) specified above corresponds to the coset \( \{(1,0),(4,1)\} \) defined at the beginning of this Section. The prime ideal \( q \subset \mathcal{O}_F \) lying over 5 having Hermite normal form \([5,1;0,1]\) lies in the ray class \((1,0)\). The Frobenius automorphism \( \sigma_q \) must be of order 6 and is therefore either equal to \( \nu \) or \( \nu^5 \). Let \( \Omega \) be the unique prime ideal in \( \mathcal{O}_K \) lying over \( q \). Using \texttt{idealfactor} again, we find that \( \nu^5(\eta) - \eta^5 \) is not divisible by \( \Omega \), proving that \( \nu = \sigma_q \), as desired. With respect to the unique prime ideal \( \mathfrak{p} \subset \mathcal{O}_K \) corresponding to the embedding of \( K \) into \( \mathbb{Q}_7 \) induced by sending \( \eta \mapsto z \) in (22) we may compute the quantities \( (\nu^j(\alpha))_{\mathfrak{p}}, 0 \leq j \leq 5 \), on the left side of the equation in part 1 of Gross’s conjecture. We easily verify that \( \text{ord}_\mathfrak{p}(\nu^j(\alpha)) = 6\zeta_T(0,\nu^j) \) for \( 0 \leq j \leq 5 \). If \( \varepsilon = \alpha/\alpha_{Gr} \), then \( |\varepsilon|_\Omega = 1 \) for every place \( \Omega \) of \( K \), which implies that \( \varepsilon \) is a root of unity in \( K \) and thus \( \varepsilon = \pm 1 \). We have \( \alpha_{\mathfrak{p}} = 1 + 6 \cdot 7 + \ldots \), and so \( \varepsilon = 1 \). \( \square \)

It is worth noting that \( \alpha_{Gr} \) is a square in \( K \) in this example and so part 2 of Gross’s conjecture holds automatically.
References


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