Rational series for multiple zeta and log gamma functions

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Abstract

We give series expansions for the Barnes multiple zeta functions in terms of rational functions whose numerators are complex-order Bernoulli polynomials, and whose denominators are linear. We also derive corresponding rational expansions for Dirichlet L-functions and multiple log gamma functions in terms of higher order Bernoulli polynomials. These expansions naturally express many of the well-known properties of these functions. As corollaries many special values of these transcendental functions are expressed as series of higher order Bernoulli numbers.

Key words: Barnes zeta functions, Hurwitz zeta function, multiple zeta functions, multiple gamma functions, Bernoulli polynomials, Dirichlet L-functions, polygamma functions

1. Introduction

Let $\zeta_r(s,a)$ denote the Barnes multiple zeta function [4, 19] of order r defined by

$$\zeta_r(s,a) = \sum_{t_1=0}^{\infty} \cdots \sum_{t_r=0}^{\infty} (a + t_1 + \cdots + t_r)^{-s} \tag{1.1}$$

for $\Re(s) > r$ and $\Re(a) > 0$, and continued meromorphically to $s \in \mathbb{C}$ with simple poles at $s = 1, 2, ..., r$. Note that $\zeta_1(s,a)$ is the Hurwitz zeta function, and $\zeta_0(s,a) = a^{-s}$ by convention. In Theorem 1 below we prove the series expansion

$$\Gamma(s)\zeta_r(s,a) = \sum_{n=0}^{\infty} \frac{(-1)^n B_n^{(n+s)}(a)}{n!(n+s-r)}, \tag{1.2}$$

where $B_n^{(z)}(x)$ is the n-th Bernoulli polynomial of order z, defined by

$$\left(\frac{t}{e^t - 1}\right)^z e^{xt} = \sum_{n=0}^{\infty} B_n^{(z)}(x) \frac{t^n}{n!}. \tag{1.3}$$

Each term in the convergent series (1.2) is a polynomial in a and a rational function of s and of the order r. We observe in section 5 below that, for nonnegative integers k, multiplying both sides by $s+k$ and taking the limit as $s \to -k$ yields the well-known values

$$\zeta_r(-k,a) = \frac{(-1)^r k!}{(r+k)!} B^{(r)}_{r+k}(a) \tag{1.4}$$

([19], eq. (3.10)) of these zeta functions at the negative integers; therefore the n-th term in the series is basically the "singular part" of $\Gamma(s)\zeta_r(s,a)$ at $s = r-n$. The series (1.2) may therefore be viewed as a polynomial Mittag-Leffler type decomposition of $\Gamma(s)\zeta_r(s,a)$ by singular parts at each pole.

The multiple log gamma function $\Psi_r(a)$ of order r is defined [4, 19] for $\Re(a) > 0$ by

$$\Psi_r(a) = \frac{\partial}{\partial s}\zeta_r(s,a)\Big|_{s=0}. \tag{1.5}$$

We observe that $\Psi_0(a) = -\log a$ following the convention $\zeta_0(s,a) = a^{-s}$. The name is derived by analogy to the case $r=1$ where we have

$$\Psi_1(a) = \log\left(\frac{\Gamma(a)}{\sqrt{2\pi}}\right) \tag{1.6}$$

([19], eq. (3.27)); its derivative $\psi(a) = \Psi_1'(a) = \Gamma'(a)/\Gamma(a)$ is called the digamma function and its higher derivatives $\psi^{(m)}(a) = \Psi_1^{(m+1)}(a)$ are called polygamma functions. In Corollary 4 below we derive the polynomial expansions

$$\Psi_r(a) = \sum_{\substack{n\geqslant 0\\ n\neq r}} \frac{(-1)^n B_n^{(n)}(a)}{n!(n-r)} + \frac{(-1)^r B_r^{(r)}(a)}{r!}\gamma + P_{r-1}(a), \tag{1.7}$$

where γ is the Euler-Mascheroni constant and $P_{r-1}(a)$ is an explicitly given polynomial of degree $r-1$, and similar expressions for the derivatives $\Psi_r^{(m)}(a)$. Again each term in the series (1.7) is a polynomial in a and a rational function of the order r.

By Lemma 2 below, the rate of convergence of the series (1.2), (1.7) is comparable to that of the series $\zeta(a+1-\varepsilon) = \sum_n n^{-(a+1-\varepsilon)}$ for any $\varepsilon > 0$; the rate of convergence is driven by $\Re(a)$ up to logarithmic factors dependent on $\Re(s)$. Our expansion of $\zeta_r(s,a)$ is influenced by recent expansions of Rubinstein [17, 18] for $\Gamma(s)$ and $\zeta_1(s,a)$; however, we avoid the beta function factors which appear in the expansion of [18] by means of the theory of Bernoulli polynomials of the second kind [5, 11, 12], while maintaining the same rate of convergence. In section 4 below we give series for many of the special values of these transcendental functions in terms of higher order Bernoulli polynomials. Then in section 5 we illustrate how many of the well-known properties of these functions are evidenced by the series (1.2), (1.7).

2. Notations and preliminaries

The Barnes multiple zeta functions [4, 19] are defined for $\Re(a) > 0$ and $\Re(s) > r$ by (1.1) and extend to a meromorphic functions of $s \in \mathbb{C}$ with simple poles at $s = 1, 2, ..., r$; by convention we have $\zeta_0(s, a) = a^{-s}$. The values of $\zeta_r(s, a)$ at the negative integers are given in terms of order r Bernoulli polynomials by (1.4). They satisfy a difference equation

$$\zeta_r(s, a) - \zeta_r(s, a + 1) = \zeta_{r-1}(s, a) \tag{2.1}$$

for positive integers r, and their a-derivatives corresponds to an s-shift

$$\frac{\partial}{\partial a}\zeta_r(s, a) = -s\zeta_r(s + 1, a) \tag{2.2}$$

([19], eq. (3.11)) for nonnegative integers r. Their s-derivatives at $s = 0$ give the multiple log gamma functions $\Psi_r(a)$ as in (1.5), which are analytic functions of a for $\Re(a) > 0$ satisfying the difference equation

$$\Psi_r(a) - \Psi_r(a + 1) = \Psi_{r-1}(a) \tag{2.3}$$

for positive integers r [4, 19]. When $r = 0$ we have $\Psi_0(a) = -\log a$, and $\Psi_1(a)$ is given by (1.6).

The gamma function $\Gamma(s)$ is defined for $\Re(s) > 0$ by the $r = 0$, $a = 1$ case of integral (3.1) and extends to a meromorphic function on $\mathbb{C}$ with simple poles at the nonpositive integers. Its logarithmic derivative

$$\frac{\Gamma'(a)}{\Gamma(a)} = \Psi_1'(a) = \psi(a) \tag{2.4}$$

is called the digamma function, the negative of whose value at $a = 1$ gives the Euler-Mascheroni constant

$$\psi(1) = \Psi_1'(1) = \Gamma'(1) = -\gamma = \lim_{n\to\infty}\left(\log n - \sum_{k=1}^{n}\frac{1}{k}\right). \tag{2.5}$$

Differentiating the functional equation $\Gamma(s+1) = s\Gamma(s)$ and letting $s \to 0$ shows that $-\gamma$ is also the constant term in the Laurent expansion of $\Gamma(s)$ about $s = 0$, namely

$$-\gamma = \lim_{s\to 0}\left(\Gamma(s) - \frac{1}{s}\right). \tag{2.6}$$

Similarly, differentiating (2.2) with respect to s and letting $s \to 0$ shows that $-\psi(a)$ is the constant term in the Laurent expansion of $\zeta_1(s, a)$ about $s = 1$, namely

$$-\psi(a) = \lim_{s\to 1}\left(\zeta_1(s, a) - \frac{1}{s - 1}\right). \tag{2.7}$$

The order z Bernoulli polynomials $B_n^{(z)}(x)$ are defined [15, 5] by (1.3); these are polynomials of degree n in x and of degree n in the order z. They satisfy a difference equation

$$B_n^{(z)}(x+1) - B_n^{(z)}(x) = nB_{n-1}^{(z-1)}(x) \tag{2.8}$$

([5], eq. (1.5)) and derivative identity

$$\frac{\partial}{\partial x} B_n^{(z)}(x) = nB_{n-1}^{(z)}(x) \tag{2.9}$$

([5], eq. (1.6)). Their dual companions are the order z Bernoulli polynomials of the second kind $b_n^{(z)}(x)$, which are defined [5, 16] by the generating function

$$\left(\frac{t}{\log(1+t)}\right)^z (1+t)^x = \sum_{n=0}^{\infty} b_n^{(z)}(x) t^n. \tag{2.10}$$

These are also polynomials of degree n in x and of degree n in the order z; Carlitz [5] originally used the notation $\beta_n^{(z)}(x)$ for $n!b_n^{(z)}(x)$, but the latter is in better agreement with the original motivation and notation of Jordan [11], and seems to have become more common [12, 1, 10, 16, 14] since it avoids confusion with the degenerate Bernoulli numbers of Carlitz. Several recent papers have also referred to the $b_n^{(z)}(x)$ as "Cauchy numbers" or "Cauchy polynomials" (cf. e.g. [13, 22]). When $z = 1$ or $x = 0$ that part of the notation is often suppressed, so that $B_n^{(z)}$ denotes $B_n^{(z)}(0)$, $b_n(x)$ denotes $b_n^{(1)}(x)$, and B_n denotes $B_n^{(1)}(0)$. The $b_n^{(z)}(x)$ satisfy a difference equation

$$b_n^{(z)}(x+1) - b_n^{(z)}(x) = b_{n-1}^{(z)}(x) \tag{2.11}$$

([5], eq. (2.4)) and derivative identity

$$\frac{\partial}{\partial x} b_n^{(z)}(x) = b_{n-1}^{(z-1)}(x) \tag{2.12}$$

([5], eq. (2.3)). The polynomials $B_n^{(z)}(x)$ and $b_n^{(z)}(x)$ may be used interchangeably; we initially derive our series for $\Gamma(s)\zeta_r(s,a)$ using a change of variable to expand an "exponential" integral (3.1) in terms of the "logarithmic" sequence $\{b_n^{(z)}(x)\}$, and then convert back to $\{B_n^{(z)}(x)\}$ by means of Carlitz's identities

$$n!b_n^{(z)}(x) = B_n^{(n-z+1)}(x+1), \qquad B_n^{(z)}(x) = n!b_n^{(n-z+1)}(x-1) \tag{2.13}$$

([5], eq. (2.11), (2.12)) in order to emphasize classical relationships such as (1.4) between $\zeta_r(s,a)$ and $B_n^{(r)}(a)$.

3. Demonstration of theorems

In this section we prove the series expansions (1.2) and (1.7) for the multiple zeta and log gamma functions.

Theorem 1. *If $\Re(a) > 0$ and $s \in \mathbb{C} \setminus \{r, r-1, ..., 1, 0, -1, -2, ...\}$ then*

$$\Gamma(s)\zeta_r(s,a) = \sum_{n=0}^{\infty} \frac{(-1)^n B_n^{(n+s)}(a)}{n!(n+s-r)}$$

for all nonnegative integers r.

Proof. For $\Re(s) > r$ and $\Re(a) > 0$ we begin with the Mellin transform integral formula

$$\Gamma(s)\zeta_r(s,a) = \int_0^\infty \frac{t^s e^{-at}}{(1-e^{-t})^r} \frac{dt}{t} \tag{3.1}$$

([19], eq. (3.2)), and make the change of variables $u = 1 - e^{-t}$, following [17]. From (2.10) we then obtain

$$\begin{aligned}
\Gamma(s)\zeta_r(s,a) &= \int_0^1 (-\log(1-u))^{s-1} u^{-r} (1-u)^{a-1}\, du \\
&= \int_0^1 \left(\frac{\log(1-u)}{-u}\right)^{s-1} (1-u)^{a-1} u^{s-r-1}\, du \\
&= \int_0^1 \sum_{n=0}^{\infty} (-1)^n b_n^{(1-s)}(a-1) u^{n+s-r-1}\, du \\
&= \sum_{n=0}^{\infty} (-1)^n b_n^{(1-s)}(a-1) \int_0^1 u^{n+s-r-1}\, du \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n b_n^{(1-s)}(a-1)}{n+s-r}, \qquad (3.2)
\end{aligned}$$

once we show that this series converges. We will then deduce the equation of the Theorem from Carlitz's identity (2.13).

The convergence of the series

$$\sum_{n=0}^{\infty} \frac{(-1)^n b_n^{(1-s)}(a-1)}{n+s-r} \tag{3.3}$$

was proved by Rubinstein [17] in the case $a = r = 1$, via the bound

$$|b_n^{(1-s)}(0)| \leqslant c_s \frac{(1+\log(n+1))^{|s|+1}}{n+1}, \tag{3.4}$$

where c_s is a constant depending only on s; that paper used the notation $\alpha_k(s)$ to denote $(-1)^k b_k^{(1-s)}(0)$. Lemma 2 below demonstrates the convergence of this series for $\Re(s) > r$ and $\Re(a) > 0$, except for the cases where a is a positive integer, by comparison with the series $\sum_n n^{-(a+1-\varepsilon)}$ for any $\varepsilon > 0$. In the case of positive integer a we observe from Carlitz's identity

$$b_n^{(1-s)}(a) = \sum_{r=0}^{n} \frac{(a)_n}{n!} b_{n-r}^{(1-s)}(0) \tag{3.5}$$

([5], eq. (2.6)), where $(a)_n = a(a-1)\cdots(a-n+1)$, that $b_n^{(1-s)}(a-1)$ is a sum of a terms of the form $b_k^{(1-s)}(0)$, with coefficients bounded independent of n; combined with Rubinstein's bound (3.4), this shows that the series (3.3) converges when $a \in \mathbb{Z}^+$, at a rate comparable to $\sum_n n^{-(2-\varepsilon)}$ for any $\varepsilon > 0$.

By means of Rubinstein's bound (3.4) and the estimate of [9, 22] (Lemma 2 below) we have demonstrated the uniform absolute convergence of the series (3.3) on compact subsets of $\Re(a) > 0$ and $\Re(s) > r$, showing that the rearrangement of integration and summation in equation (3.2) is valid for $\Re(a) > 0$ and $\Re(s) > r$. However, for fixed a with $\Re(a) > 0$ the series (3.3) converges absolutely and uniformly for s in compact subsets of $\mathbb{C} \setminus \{r, r-1, ..., 1, 0, -1, -2, ...\}$, and therefore this series provides the meromorphic continuation of $\Gamma(s)\zeta_r(s,a)$ to all $s \in \mathbb{C}$. This completes the proof of the theorem. □

The convergence of the series (3.3) for $\Re(a) > 0$ and $s \in \mathbb{C}$ may be established by the following asymptotic estimate [9, 22], together with the observation that $\log n = o(n^\varepsilon)$ for every $\varepsilon > 0$.

Lemma 2. *For $s \in \mathbb{C}$ and $a \in \mathbb{C} \setminus \mathbb{Z}^+$ we have the asymptotic estimate*

$$|b_n^{(1-s)}(a-1)| \sim \frac{1}{|(\log n)^s n^a \Gamma(1-a)|}$$

as $n \to \infty$.

Proof. This is Lemma 3.2 of [22]; see also Lemma 1 of [9]. □

The $r = 1$ case of Theorem 1 immediately gives a series representation for Dirichlet L-functions. For nontrivial Dirichlet characters χ of conductor f, the series below converges somewhat slowly, at a rate comparable to that of $\zeta(1 + (1/f)) = \sum_n n^{-(1+(1/f))}$ up to a logarithmic factor depending on s.

Corollary 3. *If χ is a nontrivial Dirichlet character of conductor f then for all $s \in \mathbb{C}$ we have*

$$L(s,\chi) = \frac{1}{\Gamma(s) f^s} \sum_{n=1}^{\infty} \frac{(-1)^n}{n!(n+s-1)} \sum_{a=1}^{f} \chi(a) B_n^{(n+s)}(a/f).$$

Proof. For nontrivial Dirichlet characters χ of conductor f we have the formula

$$L(s,\chi) := \sum_{m=1}^{\infty} \chi(m) m^{-s} = f^{-s} \sum_{a=1}^{f} \chi(a) \zeta_1(s, a/f) \tag{3.6}$$

for $\Re(s) > 0$ ([21], p. 30). The corollary then follows immediately from the $r = 1$ case of Theorem 1; the $n = 0$ term in the series vanishes since $\sum_{a=1}^{f} \chi(a) = 0$ for nontrivial characters of conductor f, leaving an everywhere-convergent expansion for the entire function $L(s,\chi)$. □

We now use Theorem 1 to derive the series expansion (1.7) for the multiple log gamma functions and their derivatives.

Corollary 4. *For $\Re(a) > 0$ the function $\Psi_r(a)$ satisfies*

$$\Psi_r(a) = \sum_{\substack{n\geqslant 0 \\ n\neq r}} \frac{(-1)^n B_n^{(n)}(a)}{n!(n-r)} + \frac{(-1)^r}{r!}\left[\gamma B_r^{(r)}(a) + \frac{\partial}{\partial s} B_r^{(r+s)}(a)\right]_{s=0}.$$

For integers m with $1 \leqslant m \leqslant r$ its m-th derivative $\Psi_r^{(m)}(a)$ satisfies

$$\Psi_r^{(m)}(a) = \sum_{\substack{n\geqslant 0 \\ n\neq r-m}} \frac{(-1)^{m+n} B_n^{(m+n)}(a)}{n!(n+m-r)} + \frac{(-1)^r}{(r-m)!}\left[\gamma B_{r-m}^{(r)}(a) + \frac{\partial}{\partial s} B_{r-m}^{(r+s-m)}(a)\right]_{s=m},$$

and for $m \geqslant r+1$ we have

$$\Psi_r^{(m)}(a) = \sum_{n=0}^{\infty} \frac{(-1)^{m+n} B_n^{(m+n)}(a)}{n!(n+m-r)}.$$

Proof. We begin the series expansion

$$\Gamma(s)\zeta_r(s,a) = \sum_{n=0}^{\infty} \frac{(-1)^n B_n^{(n+s)}(a)}{n!(n+s-r)} \tag{3.7}$$

of Theorem 1. Using the Laurent expansion (2.6) we may express the left side of (3.7) in the form

$$\Gamma(s)\zeta_r(s,a) = \frac{\zeta_r(s,a)}{s} - \gamma\zeta_r(s,a) + \sum_{n=1}^{\infty} a_n s^n \tag{3.8}$$

near $s = 0$, where $f(s) = \sum_{n=1}^{\infty} a_n s^n$ is analytic and vanishes at $s = 0$. We now subtract the $n = r$ term of the sum on the right side of (3.7) from both sides of that equation. On the left side we obtain

$$\frac{\zeta_r(s,a)}{s} - \frac{(-1)^r B_r^{(r+s)}(a)}{r!s} - \gamma\zeta_r(s,a) + f(s). \tag{3.9}$$

Using (1.4) we rewite this expression as

$$\frac{\zeta_r(s,a) - \zeta_r(0,a)}{s} + \frac{(-1)^r}{r!}\left(\frac{B_r^{(r)}(a) - B_r^{(r+s)}(a)}{s}\right) - \gamma\zeta_r(s,a) + f(s). \tag{3.10}$$

By (1.5) and (1.4), the limit as $s \to 0$ of this expression is

$$\Psi_r(a) - \frac{(-1)^r}{r!}\left[\frac{\partial}{\partial s}B_r^{(r+s)}(a)\right]_{s=0} - \gamma\zeta_r(0,a)$$
$$= \Psi_r(a) - \frac{(-1)^r}{r!}\left[\gamma B_r^{(r)}(a) + \frac{\partial}{\partial s}B_r^{(r+s)}(a)\right]_{s=0} \tag{3.11}$$

and the limit as $s \to 0$ of the right side of (3.7), with the $n = r$ term removed, is

$$\sum_{\substack{n\geqslant 0\\ n\neq r}} \frac{(-1)^n B_n^{(n)}(a)}{n!(n-r)}, \tag{3.12}$$

from which the first statement follows. The remaining statements follow by differentiating this series for $\Psi_r(a)$ using (2.9). □

Remark. The s-derivative term in the above series for $\Psi_r^{(m)}$ for $0 \leqslant m \leqslant r$ may be evaluated as

$$\left[\frac{\partial}{\partial s}B_{r-m}^{(r+s-m)}(a)\right]_{s=m} = \sum_{k=1}^{r-m}\binom{r-m}{k}\frac{(-1)^{k+1}B_k}{k}B_{r-m-k}^{(r)}(a) \tag{3.13}$$

where $B_k = B_k^{(1)}(0)$ is the ordinary Bernoulli number; this may be obtained by differentiating both sides of Ericksen's exponential representation

$$\sum_{n=0}^{\infty} B_n^{(z)}(x)\frac{t^n}{n!} = \exp\left(xt + z\sum_{n=1}^{\infty}\frac{(-1)^{n+1}B_n t^n}{n \cdot n!}\right) \tag{3.14}$$

([7], eq. (19)) with respect to z and equating coefficients of $t^n/n!$, yielding

$$\frac{\partial}{\partial z}B_n^{(z)}(x) = \sum_{k=1}^{n}\binom{n}{k}\frac{(-1)^{k+1}B_k}{k}B_{n-k}^{(z)}(x), \tag{3.15}$$

as desired.

4. Special cases

In this section we give selected cases of special values of transcendental functions which may be expressed as series of higher order Bernoulli numbers by means of Theorem 1, Corollary 3, and Corollary 4. Special cases of Theorem 1 with $r = 0$ include

Corollary 5. *For $\Re(a) > 0$ and any positive integer m we have*

$$\sum_{n=0}^{\infty}\frac{(-1)^n B_n^{(n+m)}(a)}{n!(n+m)} = \frac{(m-1)!}{a^m}$$

and for $\Re(a) > 0$ we have

$$\sum_{n=0}^{\infty} \frac{(-1)^n B_n^{(n+(1/2))}(a)}{n!(2n+1)} = \frac{1}{2}\sqrt{\frac{\pi}{a}}.$$

Equivalently we may write

$$\sum_{n=0}^{\infty} \frac{(-1)^n b_n^{(1/2)}(a-1)}{(2n+1)} = \frac{1}{2}\sqrt{\frac{\pi}{a}}.$$

Proof. The first statement is obtained by taking $r = 0$ and $s = m \in \mathbb{Z}^+$ in Theorem 1; the second statement is obtained by taking $r = 0$ and $s = 1/2$, using $\Gamma(1/2) = \sqrt{\pi}$. The last statement is a restatement of the second by means of (2.13). (We remark that we are unaware of any other known identity involving fractional order Bernoulli polynomials.) □

Special cases of Theorem 1 with $r = 1$ include

Corollary 6.

$$\sum_{n=0}^{\infty} \frac{(-1)^n B_n^{(n+2)}(1)}{n!(n+1)} = \frac{\pi^2}{6} \qquad \text{and} \qquad \sum_{n=0}^{\infty} \frac{(-1)^n B_n^{(n+4)}(1)}{n!(n+3)} = \frac{\pi^4}{15}.$$

More generally for every positive integer k we have

$$\sum_{n=0}^{\infty} \frac{(-1)^n B_n^{(n+2k)}(1)}{n!(n+2k-1)} = \frac{(-1)^{k+1}(2\pi)^{2k} B_{2k}}{4k}$$

where $B_{2k} = B_{2k}^{(1)}(0)$ is the ordinary (order 1) Bernoulli number.

Proof. These equations all follow from the $r = 1$, $a = 1$ cases of Theorem 1 by means of Euler's formula

$$\zeta(2k) = \frac{(-1)^{k+1}(2\pi)^{2k} B_{2k}}{2(2k)!} \tag{4.1}$$

for the values of $\zeta(s) = \zeta_1(s, 1)$ at the positive even integers. □

Here we give a special case arising from Corollary 3 for the nontrivial Dirichlet character of conductor 4; the first series below, consisting only of positive terms, converges slowly to the value π. Many similar series may be derived from Corollary 3 and the identities in [2].

Corollary 7.

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n \cdot n!} (B_n^{(n+1)}(1/4) - B_n^{(n+1)}(3/4)) = \pi$$

and

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n!(n+2)} (B_n^{(n+3)}(1/4) - B_n^{(n+3)}(3/4)) = 4\pi^3.$$

Proof. If χ denotes the nontrivial Dirichlet character of conductor 4, then from Corollary 3 we have

$$L(s,\chi) = \frac{1}{\Gamma(s)4^s} \sum_{n=1}^{\infty} \frac{(-1)^n}{n!(n+s-1)} (B_n^{(n+s)}(1/4) - B_n^{(n+s)}(3/4)) \tag{4.2}$$

for any $s \in \mathbb{C}$. The first result above follows by evaluating this series at $s = 1$ and noting that

$$L(1,\chi) = \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{2m-1} = \arctan 1 = \frac{\pi}{4}. \tag{4.3}$$

The second follows by evaluating (4.2) at $s = 3$ and comparing with the value $L(3,\chi) = \pi^3/32$ ([2], eq. (5.11)). □

Remark. For purposes of computation at positive integer values of s, it is easy to accelerate the convergence of series such as (4.2) by summing separately over the first several periods of the character. For any nonnegative integer N, writing

$$L(s,\chi) = \sum_{m=1}^{\infty} \chi(m)m^{-s} = \sum_{a=1}^{Nf} \chi(a)a^{-s} + \sum_{m=Nf+1}^{\infty} \chi(m)m^{-s} \tag{4.4}$$

we have

$$L(s,\chi) = \sum_{a=1}^{Nf} \chi(a)a^{-s} + f^{-s} \sum_{a=1}^{f} \chi(a)\zeta_1\left(s, N + \frac{a}{f}\right), \tag{4.5}$$

which leads to a modified Corollary 3, namely,

$$L(s,\chi) = \sum_{a=1}^{Nf} \chi(a)a^{-s} + \frac{1}{\Gamma(s)f^s} \sum_{n=1}^{\infty} \frac{(-1)^n}{n!(n+s-1)} \sum_{a=1}^{f} \chi(a)B_n^{(n+s)}(N + a/f). \tag{4.6}$$

The advantage is that the series in (4.6) then converges like $\sum_n n^{-(N+1+(1/f))}$ rather than like $\sum_n n^{-(1+(1/f))}$. In the case where χ is the nontrivial character of conductor 4 and $s = 1, 3$, for example, (4.6) with $N = 1$ reads

$$4L(1,\chi) = \frac{8}{3} + \sum_{n=1}^{\infty} \frac{(-1)^n}{n \cdot n!} (B_n^{(n+1)}(5/4) - B_n^{(n+1)}(7/4)) = \pi \tag{4.7}$$

and

$$128L(3,\chi) = 128 \cdot \frac{26}{27} + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!(n+2)} (B_n^{(n+3)}(5/4) - B_n^{(n+3)}(7/4)) = 4\pi^3, \tag{4.8}$$

which are equivalent to the series of the above Corollary but converge much more rapidly.

Among the special cases of Corollary 4 with $r = 0$ we have the following, which generalize the $a = 1$ cases which were given by Jordan ([12], p. 280, p. 277).

Corollary 8. *For $\Re(a) > 0$ we have*

$$\gamma + \log a = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} B_n^{(n)}(a)}{n \cdot n!} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} b_n(a-1)}{n}$$

and

$$\log\left(1 + \frac{1}{a}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n B_n^{(n)}(a)}{(n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n b_n(a-1)}{n+1}$$

where $b_n(x) = b_n^{(1)}(x)$ is the ordinary (order 1) Bernoulli polynomial of the second kind.

Proof. The first statement is immediate from Corollary 4 with $r = 0$. For the second, use Corollary 4 to obtain a series for $\Psi_0(a) - \Psi_0(a+1)$ and use (2.8). $\square$

Special cases of Corollary 4 with $r = 1$ include

Corollary 9. *We have a convergent series of positive terms*

$$\sum_{n=2}^{\infty} \frac{(-1)^n B_n^{(n)}(1)}{n!(n-1)} = \sum_{n=2}^{\infty} \frac{(-1)^n b_n}{n-1} = \frac{1}{2}(\gamma + 1 - \log(2\pi)).$$

Furthermore, for $\Re(a) > 0$ the digamma function $\psi(a)$ satisfies

$$\gamma + \psi(a) = \gamma + \frac{\Gamma'(a)}{\Gamma(a)} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} B_n^{(n+1)}(a)}{n \cdot n!},$$

and the polygamma function $\psi^{(m)}(a)$ satisfies

$$\psi^{(m)}(a) = \sum_{n=0}^{\infty} \frac{(-1)^{n+m+1} B_n^{(n+m+1)}(a)}{n!(n+m)}$$

for all positive integers m.

Proof. Using (3.13), the $r = 1$, $m = 0$ case of Corollary 4 reads

$$\begin{aligned}
\Psi_1(a) &= \log\left(\frac{\Gamma(a)}{\sqrt{2\pi}}\right) = \sum_{\substack{n \geqslant 0 \\ n \neq 1}} \frac{(-1)^n B_n^{(n)}(a)}{n!(n-1)} - (a - \frac{1}{2})\gamma + \frac{1}{2} \\
&= \sum_{n=2}^{\infty} \frac{(-1)^n B_n^{(n)}(a)}{n!(n-1)} - (a - \frac{1}{2})\gamma - \frac{1}{2} \qquad (4.9) \\
&= \sum_{n=2}^{\infty} \frac{(-1)^n b_n(a-1)}{n-1} - (a - \frac{1}{2})\gamma - \frac{1}{2},
\end{aligned}$$

of which the first statement is the $a = 1$ case. Since $\psi^{(m)} = \Psi_1^{(m+1)}$, the remaining statements restate the $r = 1$, $m \geqslant 1$ cases of Corollary 4; the $\psi = \Psi_1'$ case follows from the corollary by observing that $B_0^{(1)}(a) = 1$. $\square$

We remark that all of these identities may be expressed in terms of $b_n^{(z)}(x)$ by means of (2.13); in the above identities we have given the $b_n^{(z)}(x)$ forms in the cases when the order z is positive. For the purpose of computing the series, it may be preferable to express these series in their $b_n^{(z)}(x)$ form so that the order z is constant; for example, the last equation of Corollary 9 is equivalent to

$$\psi^{(m)}(a) = \sum_{n=0}^{\infty} \frac{(-1)^{n+m+1} B_n^{(n+m+1)}(a)}{n!(n+m)} = \sum_{n=0}^{\infty} \frac{(-1)^{n+m+1} b_n^{(-m)}(a-1)}{(n+m)}. \tag{4.10}$$

5. Properties of multiple zeta and log gamma functions

In this final section we illustrate briefly how several well-known properties of these functions are naturally expressed by the series in Theorem 1 and Corollary 4. The case $r = 0$, $a = 1$ of Theorem 1 reads

$$\Gamma(s) = \sum_{n=0}^{\infty} \frac{(-1)^n B_n^{(n+s)}(1)}{n!(n+s)}. \tag{5.1}$$

For nonnegative integers k, multiplying both sides by $s+k$ and taking the limit as $s \to -k$ yields

$$\lim_{s \to -k} (s+k)\Gamma(s) = \frac{(-1)^k}{k!} B_k^{(0)}(1),$$

but since it is clear from (1.3) that $B_k^{(0)}(1) = 1$ for all k, we recover the well-known residues

$$\operatorname*{Res}_{s=-k} \Gamma(s) = \frac{(-1)^k}{k!} \tag{5.2}$$

of the gamma function at its poles at the nonpositive integers. Then returning to the $r > 0$ cases of Theorem 1, multiplying both sides by $s+k$ and taking the limit as $s \to -k$ yields

$$\zeta_r(-k, a) \cdot \operatorname*{Res}_{s=-k} \Gamma(s) = \frac{(-1)^{r+k} B_{r+k}^{(r)}(a)}{(r+k)!}, \tag{5.3}$$

which gives the values (1.4) of $\zeta_r(s, a)$ at the nonpositive integers by comparison with (5.2). Similarly, when $r > 0$ and $k \in \{1, 2, ..., r\}$, multiplying both sides of Theorem 1 by $s - k$ and taking the limit as $s \to k$ yields the residues

$$\operatorname*{Res}_{s=k} \zeta_r(s, a) = \frac{(-1)^{r-k} B_{r-k}^{(r)}(a)}{(k-1)!(r-k)!} \tag{5.4}$$

([19], eq. (3.9)) of $\zeta_r(s,a)$ at each of its r poles at $s = 1, ..., r$. Furthermore, letting $s \to -k$ in Corollary 3 gives the well-known values

$$L(-k,\chi) = \frac{-f^k}{k+1}\sum_{a=1}^{f}\chi(a)B_{k+1}(a/f) = \frac{-B_{k+1,\chi}}{k+1} \tag{5.5}$$

([21], p. 31) at the nonpositive integers of Dirichlet L-function for a nontrivial character χ of conductor f in terms of the generalized Bernoulli numbers $B_{n,\chi}$, which may be defined ([21], p. 30) by

$$\sum_{a=1}^{f}\frac{\chi(a)te^{at}}{e^{ft}-1} = \sum_{n=0}^{\infty}B_{n,\chi}\frac{t^n}{n!}. \tag{5.6}$$

The fundamental difference equations (2.1) and (2.3) for $\zeta_r(s,a)$ and $\Psi_r(a)$ are readily seen to be equivalent via Theorem 1 and Corollary 4 to the difference equation (2.8) for the order z Bernoulli polynomials. The derivative-shift property (2.2) of $\zeta_r(s,a)$ is also manifest in Theorem 1 by means of the derivative relation (2.9) of the order z Bernoulli polynomials and the translation functional equation $\Gamma(s+1) = s\Gamma(s)$.

The functions $\zeta_r(s,a)$ are traditionally defined as multiple Dirichlet series (1.1) and as such require r to be a nonnegative integer; however the series in Theorem 1 makes sense for any complex r. Therefore the functions $\zeta_r(s,a)$ and $\Psi_r(a)$ could in fact be defined by means of Theorem 1 and (1.5) with arbitrary complex order r. With such a definition the difference equations (2.1) and (2.3) for $\zeta_r(s,a)$ and $\Psi_r(a)$ are still satisfied, as is the derivative-shift property (2.2) of $\zeta_r(s,a)$.

References

[1] T. Agoh and K. Dilcher, Recurrence relations for Nörlund numbers and Bernoulli numbers of the second kind, Fibonacci Quart. 48 (2010), 4 -12.

[2] E. Alkan, Values of Dirichlet L-functions, Gauss sums and trigonometric sums, Ramanujan J 26 (2011), 375-398.

[3] T. M. Apostol, *Introduction to Analytic Number Theory*, Springer–Verlag New York, 1976.

[4] E. W. Barnes, On the theory of the multiple gamma function, Trans. Cambridge Phil. Soc. 19 (1904), 374–425.

[5] L. Carlitz, A note on Bernoulli and Euler polynomials of the second kind, Scripta Math. 25 (1961), 323-330.

[6] H. Cohen, *Number theory. Volume II: Analytic and modern tools*, Springer-Verlag, New York, 2007.

[7] L. Ericksen, Sum of powers from uniform distribution moments, J. Stat. Plan. Inference 101 (2002), 57-79.

[8] P. Flajolet and A. Oldyzko, Singularity analysis of generating functions, SIAM J. Disc. Math. 3 (1990), 216-240.

[9] P. Flajolet, E. Fusy, X. Gourdon, D. Panario and N. Pouyanne, A hybrid of Darboux's method and singularity analysis in combinatorial asymptotics, Electron. J. Combin. 13 (2006), 1-35.

[10] F. T. Howard, Congruences and recurrences for Bernoulli numbers of higher order, Fibonacci Quart. 32 (1994), 316 - 328.

[11] C. Jordan, Polynômes analogous aux polynômes de Bernoulli, Acta Scient. Math. 4 (1929), 130-150.

[12] C. Jordan, *Calculus of finite differences*, Chelsea, New York, 1965.

[13] D. Merlini, R. Sprugnoli, and M. Verri, The Cauchy numbers, Discrete Math 306 (2006), 1906-1920.

[14] Nemes, G., An asymptotic expansion for the Bernoulli numbers of the second kind, J. Integer Seq. 14 (2011), article 11.4.8, 6 pp.

[15] N. E. Nörlund, *Differenzenrechnung*, Springer, 1924.

[16] S. Roman, *The Umbral Calculus*, Academic Press, New York, 1984.

[17] M. O. Rubinstein, Identities for the Riemann zeta function, Ramanujan J. 27 (2012), 29-42.

[18] M. O. Rubinstein, Identities for the Hurwitz zeta function, gamma function, and L-functions, arXiv 1206:1992v1 (2012).

[19] S. N. M. Ruijsenaars, On Barnes' multiple zeta and gamma functions, Adv. Math. 156 (2000), 107–132.

[20] I. Vardi, Determinants of Laplacians and multiple gamma functions, SIAM J. Math. Anal. 19 (1988), 493-507.

[21] L. Washington, *Introduction to Cyclotomic Fields*, Springer-Verlag, New York, 1982.

[22] F. Zhao, Some properties of a new class of generalized Cauchy numbers, Online J. Analytic Combin. 6 (2011), article 5 (electronic), 11pp.