Symmetries of Bernoulli polynomial series and Arakawa-Kaneko zeta functions

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Abstract

The Arakawa-Kaneko zeta functions interpolate the poly-Bernoulli numbers at the negative integers and their values at positive integers are connected to multiple zeta values. We give everywhere-convergent series expansions of these functions in terms of Bernoulli polynomials and Dirichlet series related to harmonic numbers, exhibiting their explicit analytic continuation. Differentiating the Barnes multiple zeta functions of order r with respect to their order produces Dirichlet series attached to Bernoulli polynomials. These series are invariant under an involution in which the order of the derivative is dual to the value of the first variable and the order of the zeta function is dual to the value of the second variable. This symmetry relation generalizes duality relations of Euler sums, and is featured in our series expansions of Arakawa-Kaneko zeta values.

Keywords: Barnes zeta functions, Hurwitz zeta function, Arakawa-Kaneko zeta functions, poly-Bernoulli polynomials, Bernoulli polynomials, Euler sums

1. Introduction

In the early twentieth century E.W. Barnes [4] introduced a general class of multiple zeta and gamma functions and developed a comprehensive analytic and algebraic theory surrounding them. For each nonnegative integer r he defined zeta functions of order r as functions of complex variables dependent on r parameters. In the case of equal parameters these zeta functions may be viewed as analytic functions of the order r [23] and their derivatives with respect to their order r are expressible as Dirichlet series attached to complex-order Bernoulli polynomials and generalized harmonic numbers (Theorem 1 below). In section 3 we show that these series representations exhibit a perfect symmetry which may be expressed in the form

$$\Gamma(j)\sum_{m=0}^{\infty}\frac{(-1)^m B_m^{(m+k)}(1-r)}{m!(m+k-t)^j} = \Gamma(k)\sum_{n=0}^{\infty}\frac{(-1)^n B_n^{(n+j)}(1-t)}{n!(n+j-r)^k} \tag{1.1}$$

(Corollary 2 below), where $B_n^{(z)}(x)$ denotes the n-th Bernoulli polynomial of order z. This symmetry generalizes duality relations of harmonic number series.

In 1999 Arakawa and Kaneko [3] defined zeta functions $\xi_k(s)$ which interpolate the poly-Bernoulli numbers of order k at the negative integers, and whose values at positive integers are connected to multiple zeta values. In section 4 we give an everywhere-convergent series expansion for the Arakawa-Kaneko zeta functions in terms of complex-order Bernoulli polynomials and Dirichlet series related to harmonic numbers. We analyze the log gamma and digamma functions associated to these zeta functions in section 5, and in section 6 we give series representations for the values at positive integers in terms of derivatives of complex order zeta functions with respect to their order, in forms which may be expressed in terms of generalized harmonic number series.

2. Notations and preliminaries

The order z Bernoulli polynomials $B_n^{(z)}(x)$ are defined [17, 6] by

$$\left(\frac{t}{e^t-1}\right)^z e^{xt} = \sum_{n=0}^{\infty} B_n^{(z)}(x)\frac{t^n}{n!}; \tag{2.1}$$

these are polynomials of degree n in x and of degree n in the order z. They satisfy a difference equation

$$B_n^{(z)}(x+1) - B_n^{(z)}(x) = nB_{n-1}^{(z-1)}(x) \tag{2.2}$$

([6], eq. (1.5)) and derivative identity

$$\frac{\partial}{\partial x}B_n^{(z)}(x) = nB_{n-1}^{(z)}(x) \tag{2.3}$$

([6], eq. (1.6)). Their dual companions are the order z Bernoulli polynomials of the second kind $b_n^{(z)}(x)$, which are defined [6, 18] by the generating function

$$\left(\frac{t}{\log(1+t)}\right)^z (1+t)^x = \sum_{n=0}^{\infty} b_n^{(z)}(x)t^n. \tag{2.4}$$

These are also polynomials of degree n in x and of degree n in the order z. Several recent papers have also referred to the $b_n^{(z)}(x)$ as "Cauchy numbers" or "Cauchy polynomials" (cf. e.g. [16, 24]). When $z = 1$ or $x = 0$ that part of the notation is often suppressed, so that $B_n^{(z)}$ denotes $B_n^{(z)}(0)$, $b_n(x)$ denotes $b_n^{(1)}(x)$, and B_n denotes $B_n^{(1)}(0)$. The $b_n^{(z)}(x)$ satisfy a difference equation

$$b_n^{(z)}(x+1) - b_n^{(z)}(x) = b_{n-1}^{(z)}(x) \tag{2.5}$$

([6], eq. (2.4)) and derivative identity

$$\frac{\partial}{\partial x}b_n^{(z)}(x) = b_{n-1}^{(z-1)}(x) \tag{2.6}$$

([6], eq. (2.3)). The polynomials $B_n^{(z)}(x)$ and $b_n^{(z)}(x)$ may be used interchangeably; each may be converted into the other form by means of Carlitz's identities

$$n!b_n^{(z)}(x) = B_n^{(n-z+1)}(x+1), \qquad B_n^{(z)}(x) = n!b_n^{(n-z+1)}(x-1) \tag{2.7}$$

([6], eq. (2.11), (2.12)). For integer values of z the Bernoulli polynomials also include Stirling numbers as special cases (cf. [11], §2); specifically, for nonnegative integers m the Stirling numbers of the first kind may be defined by

$$s(n+m,m) := \binom{n+m-1}{n} B_n^{(n+m)} \tag{2.8}$$

by means of (2.4) and (2.7), and the weighted Stirling numbers of the second kind may be defined by

$$S(n+m,m\,|\,a) := \binom{n+m}{m} B_n^{(-m)}(a) \tag{2.9}$$

by means of (2.1), with $S(n,m) := S(n,m\,|\,0)$ being the usual Stirling numbers of the second kind. In [23] we observed that for $s,a \in \mathbb{C}$ we have

$$\left|\frac{B_n^{(n+s)}(a)}{n!}\right| = \begin{cases} O(n^{-a+\varepsilon}), & \text{if } a \notin \mathbb{Z}^+ \\ O(n^{-1+\varepsilon}), & \text{if } a \in \mathbb{Z}^+ \end{cases} \tag{2.10}$$

as $n \to \infty$ for any $\varepsilon > 0$, using results from [19, 24, 10].

The *poly-Bernoulli polynomials* $\mathbb{B}_n^{(k)}(a)$ of order k are defined [8] by means of the generating function

$$\frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}} e^{-at} = \sum_{n=0}^{\infty} \mathbb{B}_n^{(k)}(a) \frac{t^n}{n!} \tag{2.11}$$

where

$$\mathrm{Li}_k(z) = \sum_{m=1}^{\infty} \frac{z^m}{m^k} \tag{2.12}$$

is the order k polylogarithm function. We remark that the order k has generally been taken to be an arbitrary integer, although in this paper we will consider k to be a complex number with positive real part. The polynomial $\mathbb{B}_n^{(k)}(a)$ is a polynomial in a of degree n, and an analytic function of the order k. For $a=0$ the values $\mathbb{B}_n^{(k)}(0) = \mathbb{B}_n^{(k)}$ are the poly-Bernoulli numbers and for $a=1$ the values $\mathbb{B}_n^{(k)}(1) = C_n^{(k)}$ are the modified poly-Bernoulli numbers of Arakawa and Kaneko [3, 15]. Comparison of (2.11) and (2.1) shows that the Bernoulli and poly-Bernoulli polynomials are related by

$$\mathbb{B}_n^{(1)}(a) = (-1)^n B_n^{(1)}(a). \tag{2.13}$$

The generalized harmonic numbers $H_n^{(m)}$ are defined by

$$H_n^{(m)} = \sum_{j=1}^{n} \frac{1}{j^m} \tag{2.14}$$

with $H_n := H_n^{(1)}$ being the usual harmonic numbers. For positive integers n and k the numbers $B_n^{(n+k)}$ and $B_n^{(n+k)}(1)$ can always be expressed in terms of generalized harmonic numbers; this follows from (2.8), the relation

$$\frac{B_n^{(n+k)}}{n!} = (-1)^n w(n+k, k-1) \tag{2.15}$$

([1], eq. (3)), where $w(n,m)$ satisfies the recursion

$$w(n,m) = \sum_{k=0}^{m-1} m! \binom{k-m}{m} H_{n-1}^{(k+1)} w(n, m-k-1) \tag{2.16}$$

with $w(n,0) = 1$, and the relation

$$B_n^{(n+k+1)}(1) = \frac{k}{n+k} B_n^{(n+k)} \tag{2.17}$$

which follows from the generating function (2.4) and (2.7). In addition to (3.6) we list as examples

$$\frac{B_n^{(n+2)}}{n!} = (-1)^n H_{n+1}; \qquad \frac{B_n^{(n+2)}(1)}{n!} = \frac{(-1)^n}{n+1}; \tag{2.18}$$

$$\frac{B_n^{(n+3)}}{n!} = (-1)^n \left(H_{n+2}^2 - H_{n+2}^{(2)} \right); \qquad \frac{B_n^{(n+3)}(1)}{n!} = \frac{2(-1)^n H_{n+1}}{n+2}; \tag{2.19}$$

$$\frac{B_n^{(n+4)}}{n!} = (-1)^n \left(H_{n+3}^3 - 3H_{n+3}^{(2)} H_{n+3} + 2H_{n+3}^{(3)} \right). \tag{2.20}$$

A series of the form

$$\sum_{n=1}^{\infty} \frac{H_n^{(e_1)} H_n^{(e_2)} \cdots H_n^{(e_p)}}{n^q} \tag{2.21}$$

with $q > 1$ is called an *Euler sum* of weight $e_1 + e_2 + \cdots + e_p + q$. Such series may generally be expressed in terms of values of Arakawa-Kaneko zeta functions (cf. [1, 9, 8]); in this paper we give generalizations of several such relations to sums with denominator $(n+a)^q$.

3. Complex order zeta functions and associated Dirichlet series

For nonnegative integers r let $\zeta_r(s,a)$ denote the Barnes multiple zeta function [4, 21] of order r defined by

$$\zeta_r(s,a) = \sum_{t_1=0}^{\infty} \cdots \sum_{t_r=0}^{\infty} (a + t_1 + \cdots + t_r)^{-s} \tag{3.1}$$

for $\Re(s) > r$ and $\Re(a) > 0$, and continued meromorphically to $s \in \mathbb{C}$ with simple poles at $s = 1, 2, ..., r$. Note that $\zeta_1(s,a)$ is the Hurwitz zeta function, and $\zeta_0(s,a) = a^{-s}$ by convention. For $\Re(a) > 0$ we will consider $\zeta_r(s,a)$ as an analytic function of its order r for arbitrary $r \in \mathbb{C}$, using the equivalent definitions

$$\zeta_r(s,a) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^s e^{-at}}{(1-e^{-t})^r} \frac{dt}{t} \tag{3.2}$$

$$= \sum_{m=0}^\infty \binom{m+r-1}{m} (m+a)^{-s} \tag{3.3}$$

$$= \frac{1}{\Gamma(s)} \sum_{n=0}^\infty \frac{(-1)^n B_n^{(n+s)}(a)}{n!(n+s-r)} \tag{3.4}$$

[21, 22, 23]; For nonnegative integer r and $\Re(a) > 0$ the first two are valid for $\Re(s) > r$ and the third is valid for all $s \in \mathbb{C}$, with poles at $s \in \{1,2,...,r\}$. For fixed s,a with $\Re(a) > 0$ the first two expressions define analytic functions of r for $\Re(r) < \Re(s)$; the third is a meromorphic function of r with poles wherever $r - s$ is a nonnegative integer. The equality of (3.2) and (3.3) for arbitrary r is seen from

$$(1-e^{-t})^{-r} = \sum_{m=0}^\infty \binom{-r}{m} (-1)^m e^{-mt} = \sum_{m=0}^\infty \binom{r+m-1}{m} e^{-mt} \tag{3.5}$$

and the well-known

$$\int_0^\infty t^s e^{-bt} \frac{dt}{t} = b^{-s}\Gamma(s)$$

for $\Re(b) > 0$ ([21], eq. (2.7)); the equality of (3.2) and (3.4) for general r was observed in ([23], Theorem 1; §5). The following theorem gives two equivalent expressions for their derivatives with respect to r:

Theorem 1. *For all nonnegative integers j the j-th partial derivative of $\zeta_r(s,a)$ with respect to the order r satisfies*

$$D_r^j \zeta_r(s,a) = \sum_{m=j}^\infty \frac{(-1)^{m-j} B_{m-j}^{(m+1)}(1-r)}{(m-j)!(m+a)^s} = \frac{j!}{\Gamma(s)} \sum_{n=0}^\infty \frac{(-1)^n B_n^{(n+s)}(a)}{n!(n+s-r)^{j+1}}$$

for $\Re(a) > -j$ and $\Re(s) > \Re(r)$. As a function of r the second series is analytic for all $r \in \mathbb{C}$ except for poles where $r - s$ is a nonnegative integer, providing a meromorphic continuation of the first series. As a function of s the second series is analytic for all $s \in \mathbb{C}$ except for poles where $s - r$ is a nonpositive integer, providing a meromorphic continuation of the first series.

Proof. From (2.7) and the generating function (2.4) we have

$$\frac{B_n^{(n+1)}(a)}{n!} = \binom{a-1}{n} = (-1)^n \binom{n-a}{n} \tag{3.6}$$

so that the $j = 0$ case of the theorem is a restatement of (3.3) and (3.4). For $j > 0$, differentiate termwise using (2.3) and observe that the abscissa of convergence decreases with each successive differentiation, providing analytic continuation of the derivatives to successively larger domains. $\square$

In the above theorem the first series for $D_r^j \zeta_r(s,a)$ converges at a rate comparable to $\sum_m m^{r-s-1+\varepsilon}$ when $1-r \notin \mathbb{Z}^+$ and to $\sum_m m^{-1-s+\varepsilon}$ when $1-r \in \mathbb{Z}^+$; the second series converges at a rate comparable to $\sum_n n^{-a-j-1+\varepsilon}$ when $a \notin \mathbb{Z}^+$ and to $\sum_n n^{-2-j+\varepsilon}$ when $a \in \mathbb{Z}^+$. By means of (2.15) and (2.17) these expressions for $D_r^j \zeta_r(s,a)$ may always be expressed as harmonic number series when $r = 0$ or $r = 1$. If we also have $a \in \{0,1\}$ and $s \in \mathbb{Z}^+$ these expressions yield Euler sums of weight $j+s$. As examples with $r = 0$ we have

$$D_r \zeta_r(s,a)\bigg|_{r=0} = \sum_{m=1}^{\infty} \frac{1}{m(m+a)^s} = \frac{1}{\Gamma(s)} \sum_{n=0}^{\infty} \frac{(-1)^n B_n^{(n+s)}(a)}{n!(n+s)^2} \tag{3.7}$$

for $\Re(a) > -1$, and

$$D_r^2 \zeta_r(s,a)\bigg|_{r=0} = 2 \sum_{m=2}^{\infty} \frac{H_{m-1}}{m(m+a)^s} = \frac{2}{\Gamma(s)} \sum_{n=0}^{\infty} \frac{(-1)^n B_n^{(n+s)}(a)}{n!(n+s)^3} \tag{3.8}$$

for $\Re(a) > -2$. As examples with $r = 1$ we have

$$D_r \zeta_r(s,a)\bigg|_{r=1} = \sum_{m=1}^{\infty} \frac{H_m}{(m+a)^s} = \frac{1}{\Gamma(s)} \sum_{n=0}^{\infty} \frac{(-1)^n B_n^{(n+s)}(a)}{n!(n+s-1)^2} \tag{3.9}$$

for $\Re(a) > -1$ and

$$D_r^2 \zeta_r(s,a)\bigg|_{r=1} = \sum_{m=2}^{\infty} \frac{H_m^2 - H_m^{(2)}}{(m+a)^s} = \frac{2}{\Gamma(s)} \sum_{n=0}^{\infty} \frac{(-1)^n B_n^{(n+s)}(a)}{n!(n+s-1)^3} \tag{3.10}$$

for $\Re(a) > -2$.

As a corollary we have the following perfectly symmetric series identity:

Corollary 2. *For positive integers j, k and complex variables r, t with $\Re(r) < j$ and $\Re(t) < k$ we have*

$$\Gamma(k) D_t^{j-1} \zeta_t(k, 1-r) = \Gamma(j) D_r^{k-1} \zeta_r(j, 1-t),$$

that is,

$$\Gamma(j) \sum_{m=0}^{\infty} \frac{(-1)^m B_m^{(m+k)}(1-r)}{m!(m+k-t)^j} = \Gamma(k) \sum_{n=0}^{\infty} \frac{(-1)^n B_n^{(n+j)}(1-t)}{n!(n+j-r)^k}.$$

We remark that this identity bears a striking resemblance to the identity $\mathbb{B}_n^{(-k)} = \mathbb{B}_k^{(-n)}$ for negative-order poly-Bernoulli numbers ([14], Theorem 2) by virtue of (4.13), although it also includes an extra facet of symmetry in that it is invariant under $(j,k,r,t) \mapsto (k,j,t,r)$. The $r=1$ case of Theorem 1 will be a primary focus of this paper, since the series $D_r^j \zeta_1(s,a)$ appear in our series expansions (Theorem 10) for the Arakawa-Kaneko zeta functions $Z_k(s,a)$. Using (2.15) and (2.17), this corollary also generalizes relations of duality between Euler sums; for example the special case

$$6D_t\zeta_t(4,0)\Big|_{t=1} = D_r^3\zeta_r(2,0)\Big|_{r=1} \tag{3.11}$$

of the above corollary may be written explicitly as

$$\sum_{m=0}^{\infty} \frac{H_{m+3}^3 - 3H_{m+3}^{(2)}H_{m+3} + 2H_{m+3}^{(3)}}{(m+3)^2} = 6\sum_{n=0}^{\infty} \frac{H_{n+1}}{(n+1)^4}. \tag{3.12}$$

Both these sums have the value $12Z_3(3,1)$ (equation (6.1) below).

4. Arakawa-Kaneko zeta functions

Coppo and Candelpergher [8] defined functions $\xi_k(s,x)$ extending the functions $\xi_k(s) := \xi_k(s,1)$ defined by Arakawa and Kaneko [3] and showed how their values at positive integers s are connected to generalized harmonic number series. The functions $\xi_k(s)$ and $\xi_k(s,x)$ are generalized Barnes zeta functions of order zero (cf. [21], §2) associated to the function

$$f(t) = \frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}} \tag{4.1}$$

which generates the poly-Bernoulli numbers. In this paper we will work with the associated first-order zeta function $Z_k(s,a)$ which is defined for $\Re(s) > 1$, $\Re(k) > 0$, and $\Re(a) > 0$ by

$$\Gamma(s)Z_k(s,a) = \int_0^\infty t^{s-2}e^{-at}f(t)\,dt \tag{4.2}$$

and extends to a meromorphic function of $s \in \mathbb{C}$ with a single simple pole at $s=1$ with residue 1, and satisfying

$$Z_k(-m,a) = \frac{(-1)^m \mathbb{B}_{m+1}^{(k)}(a)}{m+1} \tag{4.3}$$

for nonnegative integers m ([21], Prop. 2.2). These functions are related by

$$\xi_k(s,a) = sZ_k(s+1,a) = -\frac{\partial}{\partial a}Z_k(s,a). \tag{4.4}$$

In order to facilitate comparisons, we prefer to work with $Z_k(s,a)$ as a natural generalization of the Hurwitz zeta function $\zeta(s,a) = Z_1(s,a)$, as it is also a first-order zeta function.

We prove the everywhere-convergent series expansion

Theorem 3. *If $\Re(a) > 0$, $\Re(k) > 0$, and $\Re(a) + \Re(k) > 1$, then for $s \in \mathbb{C} \setminus \{1, 0, -1, -2, ...\}$ we have*

$$\Gamma(s) Z_k(s,a) = \sum_{n=0}^{\infty} \frac{(-1)^n B_n^{(n+s-1)}(a)}{n!} H(k, n+s-2)$$

where $H(k,z)$ denotes the Dirichlet series defined by

$$H(k,z) = \sum_{m=1}^{\infty} \frac{1}{(m+z)m^k}$$

for $\Re(k) > 0$ and $z \in \mathbb{C} \setminus \{-1, -2, ...\}$.

Proof. For $\Re(s) > 1$ and $\Re(a) > 0$ we begin with the definition (4.2) and make the change of variables $u = 1 - e^{-t}$, following [19, 20]. From (2.4) we then obtain

$$\begin{aligned}
\Gamma(s) Z_k(s,a) &= \int_0^1 (-\log(1-u))^{s-2} (1-u)^{a-1} \frac{\mathrm{Li}_k(u)}{u}\, du \\
&= \int_0^1 \left(\frac{\log(1-u)}{-u} \right)^{s-2} (1-u)^{a-1} u^{s-3} \sum_{m=1}^{\infty} \frac{u^m}{m^k}\, du \\
&= \int_0^1 \sum_{n=0}^{\infty} (-1)^n b_n^{(2-s)}(a-1) u^n \sum_{m=0}^{\infty} \frac{u^{m+s-3}}{m^k}\, du \\
&= \sum_{n=0}^{\infty} (-1)^n b_n^{(2-s)}(a-1) \sum_{m=0}^{\infty} m^{-k} \int_0^1 u^{m+n+s-3}\, du \\
&= \sum_{n=0}^{\infty} (-1)^n b_n^{(2-s)}(a-1) H(k, n+s-2), \qquad (4.5)
\end{aligned}$$

provided this latter series converges. By means of (2.10) and Lemma 4 below this latter series converges absolutely and uniformly on compact subsets of $\mathbb{C} \setminus \{1, 0, -1, -2, ...\}$ under the stated assumptions on a and k, justifying the rearrangement of integration and summation in (4.5) and providing the analytic continuation of $Z_k(s,a)$ to $\mathbb{C} \setminus \{1, 0, -1, -2, ...\}$. The series may then be written in the form given in the statement of the theorem by means of (2.7). □

This theorem provides an explicit analytic continuation of the integral representation (4.2) from $\Re(s) > 1$ to all of $\mathbb{C}$. The convergence of the series in the above theorem may be established by means of (2.10) and the following asymptotic estimates for the function $H(k,z)$.

Lemma 4. *For $0 < \Re(k) < 1$ we have*

$$|H(k,z)| = O(|z|^{-k}) \qquad \text{as} \qquad \Re(z) \to \infty.$$

Furthermore

$$|H(1,z)| \sim \left|\frac{\log z}{z}\right| \qquad \text{as} \qquad \Re(z) \to \infty,$$

and for $\Re(k) > 1$ *we have*

$$|H(k,z)| \sim \left|\frac{\zeta(k)}{z}\right| \qquad \text{as} \qquad \Re(z) \to \infty.$$

Proof. We first observe that if $\Re(z) \geqslant x > 0$ and $\Re(k) \geqslant c > 0$ then $|H(k,z)| \leqslant H(c,x)$, so for purposes of estimation it suffices to assume $k, z \in \mathbb{R}^+$. For $0 < c < 1$ and $x > 0$ we have

$$\begin{aligned}
H(c,x) &= \sum_{m=1}^{\infty} \frac{1}{(m+x)m^c} \leqslant \int_0^\infty \frac{dt}{(t+x)t^c} \\
&= \frac{x^{-c}}{1-c} \int_0^\infty \frac{du}{1+u^b} \qquad \left(\text{where} \quad b = \frac{1}{1-c}\right) \\
&< \frac{x^{-c}}{1-c} \left(1 + \int_1^\infty \frac{du}{u^b}\right) \\
&= \frac{x^{-c}}{1-c} \left(1 + \frac{1}{b-1}\right) = O(x^{-c})
\end{aligned} \tag{4.6}$$

via the change of variables $u = (t/x)^{1-c}$, giving the first statement.

For positive integers n it is easily seen that

$$H(1,n) = \frac{H_n}{n}, \tag{4.7}$$

and more generally for $\Re(z) > -1$ we have

$$H(1,z) = \frac{\psi(z+1) - \psi(1)}{z} \tag{4.8}$$

where $\psi(z) = \Gamma'(z)/\Gamma(z)$ denotes the digamma function, giving the estimate for $H(1,z)$. Finally, for $\Re(k) > 0$ we have the recurrence

$$zH(k+1,z) = \zeta(k+1) - H(k,z) \tag{4.9}$$

immediately from the definition of $H(k,z)$, so the final statement follows directly from the previous cases. $\square$

Obviously $H(k,0) = \zeta(k+1)$ and

$$\sum_{n=1}^{\infty} \frac{H(k,n)}{n} = \sum_{m=1}^{\infty} \frac{H_m}{m^{k+1}} \tag{4.10}$$

for $\Re(k) > 0$. For positive integers k, $H(k,z)$ extends to a meromorphic function of $z \in \mathbb{C}$ with poles at the negative integers, satisfying

$$H(k,z) = (-1)^{k+1}\frac{\psi(z+1)-\psi(1)}{z^k} + \sum_{j=2}^{k}(-1)^{k+j}\frac{\zeta(j)}{z^{k+1-j}} \tag{4.11}$$

for $\Re(z) > -1$; and for positive integers n we have

$$H(k,n) = (-1)^{k+1}\frac{H_n}{n^k} + \sum_{j=2}^{k}(-1)^{k+j}\frac{\zeta(j)}{n^{k+1-j}}; \tag{4.12}$$

these follow directly from (4.8) and the recurrence (4.9).

For nonnegative integers j, multiplying both sides of Theorem 3 by $s+j$ and taking the limit as $s \to -j$ yields

$$\frac{\mathbb{B}_{j+1}^{(k)}(a)}{(j+1)!} = \sum_{m=0}^{j+1}\frac{(-1)^{j+1-m}B_{j+1-m}^{(-m)}(a)}{(j+1-m)!(m+1)^k} \tag{4.13}$$

by means of (4.3) and the easily established residues

$$\operatorname*{Res}_{s=-j} H(k, n+s-2) = \frac{1}{(j+2-n)^k}. \tag{4.14}$$

By means of (2.9) this may be put in the form

$$\mathbb{B}_{j+1}^{(k)}(a) = \sum_{m=0}^{j+1}\frac{(-1)^{j+1-m}m!S(j+1,m\,|\,a)}{(m+1)^k} \tag{4.15}$$

in agreement with ([14], Theorem 1). By means of (4.8) the $k=1$ case of Theorem 3 may also be put in the form

$$\sum_{n=0}^{\infty}\frac{(-1)^n B_n^{(n+s)}(a)\psi(n+s)}{n!(n+s-1)} = \Gamma(s)\left(s\zeta(s+1,a) - \gamma\zeta(s,a)\right) \tag{4.16}$$

for $\Re(a) > 0$.

5. Log gamma and digamma functions

We define an Arakawa-Kaneko "log gamma" function $L\Gamma(k,a)$ to be the derivative with respect to s of $Z_k(s,a)$ at $s=0$, namely

$$L\Gamma(k,a) = \frac{\partial}{\partial s}Z_k(s,a)\Big|_{s=0} \tag{5.1}$$

(cf. [21], eq. (2.14)). This function may be viewed as the logarithm of a generalized gamma function associated to the polylogarithm function; the name is in analogy to the classical case $k=1$ where we have

$$L\Gamma(1,a) = \Psi(a) := \log\left(\frac{\Gamma(a)}{\sqrt{2\pi}}\right) \tag{5.2}$$

([21], eq. (3.27)). The function $L\Gamma(k,a)$ also has the integral representation

$$L\Gamma(k,a) = \int_0^\infty \left(e^{-at} f(t) - 1 - te^{-t}\mathbb{B}_1^{(k)}(a)\right) \frac{dt}{t^2} \tag{5.3}$$

([21], eq. (2.35)) where $f(t)$ is as in (4.1).

Theorem 5. *For $\Re(a) > 0$, $\Re(k) > 0$, and $\Re(a) + \Re(k) > 1$ we have*

$$\begin{aligned} L\Gamma(k,a) &= \gamma\mathbb{B}_1^{(k)}(a) - \frac{1}{2} + \sum_{n=1}^\infty 2^n(\zeta(k+n) - 1 - 2^{-k}) \\ &\quad -a\sum_{n=1}^\infty(\zeta(k+n)-1) + \sum_{n=2}^\infty \frac{(-1)^n B_n^{(n-1)}(a)}{n!} H(k, n-2) \end{aligned}$$

where γ denotes the Euler-Mascheroni constant. For positive integers k this may be written in the form

$$\begin{aligned} L\Gamma(k,a) &= \gamma\mathbb{B}_1^{(k)}(a) - ak - \sum_{j=2}^{k} \zeta(j)\mathbb{B}_1^{(k+1-j)}(a) \\ &\quad + \frac{1}{2} + \frac{k-2}{2^{k+1}} + \sum_{n=2}^\infty \frac{(-1)^n B_n^{(n-1)}(a)}{n!} H(k, n-2). \end{aligned}$$

Proof. Write the equation of Theorem 3 in the form

$$\begin{aligned} &\Gamma(s)Z_k(s,a) - H(k,s-2) + B_1^{(s)}(a)H(k,s-1) \\ &\qquad = \sum_{n=2}^\infty \frac{(-1)^n B_n^{(n+s-1)}(a)}{n!} H(k, n+s-2) \end{aligned} \tag{5.4}$$

so that the right side is analytic at $s = 0$. Near $s = 0$, $\Gamma(s)$ has the Laurent expansion

$$\Gamma(s) = \frac{1}{s} - \gamma + h(s) \tag{5.5}$$

where $h(s)$ is analytic and vanishes at $s = 0$; therefore the left side of (5.4) may be written as

$$\frac{Z_k(s,a)}{s} - \gamma Z_k(s,a) - H(k,s-2) + B_1^{(s)}(a)H(k,s-1) + g(s) \tag{5.6}$$

where $g(s)$ is analytic and vanishes at $s = 0$. By means of (4.3) this may be written as

$$\frac{Z_k(s,a) - Z_k(0,a)}{s} + \frac{\mathbb{B}_1^{(k)}(a)}{s} - \gamma Z_k(s,a) - H(k,s-2) + B_1^{(s)}(a)H(k,s-1) + g(s). \tag{5.7}$$

As s approaches zero, the analytic terms in (5.7) have limit

$$\frac{Z_k(s,a) - Z_k(0,a)}{s} - \gamma Z_k(s,a) + g(s) \;\to\; L\Gamma(k,a) - \gamma \mathbb{B}_1^{(k)}(a) + 0 \tag{5.8}$$

by (5.1) and (4.3). The singular parts of the remaining terms in (5.7) cancel, yielding the limit

$$\begin{aligned} &\frac{2^{-k} - a}{s} - \sum_{m=1}^{\infty} \frac{1}{m^k(m+s-2)} + \left(a - \frac{s}{2}\right) \sum_{m=1}^{\infty} \frac{1}{m^k(m+s-1)} \\ &\qquad \to \frac{1}{2} - \sum_{m=3}^{\infty} \frac{1}{m^k(m-2)} + a \sum_{m=2}^{\infty} \frac{1}{m^k(m-1)} \end{aligned} \tag{5.9}$$

via (2.1), (2.14), and (4.3). By expanding $(1 - m^{-1})^{-1}$ and $(1 - 2m^{-1})^{-1}$ as geometric series and rearranging, these two sums may be rewritten in the forms

$$\sum_{m=2}^{\infty} \frac{1}{m^k(m-1)} = \sum_{n=1}^{\infty} (\zeta(k+n) - 1) = k - \sum_{j=2}^{k} \zeta(j) \tag{5.10}$$

and

$$\sum_{m=3}^{\infty} \frac{1}{m^k(m-2)} = \sum_{n=1}^{\infty} 2^n (\zeta(k+n) - 1 - 2^{-k}) = 1 + \frac{k-2}{2^{k+1}} - \sum_{j=2}^{k} \frac{\zeta(j)}{2^{k+1-j}}, \tag{5.11}$$

where in each case the first equality holds for $\Re(k) > 0$ and the second holds for $k \in \mathbb{Z}^+$. The limit of the right side of (5.4) as $s \to 0$ is

$$\sum_{n=2}^{\infty} \frac{(-1)^n B_n^{(n+s-1)}(a)}{n!} H(k, n+s-2) \;\to\; \sum_{n=2}^{\infty} \frac{(-1)^n B_n^{(n-1)}(a)}{n!} H(k, n-2) \tag{5.12}$$

from which the first statement follows. The second form for $k \in \mathbb{Z}^+$ is obtained by observing that $\mathbb{B}_1^{(k)}(a) = 2^{-k} - a$, which is the $j = 0$ case of (4.13). $\square$

The $k = 1$ case of this theorem gives a series expansion for $\Psi(a)$ involving Bernoulli polynomials which is different from that given in [23], including an unusual series involving harmonic numbers and Bernoulli numbers of the second kind:

Corollary 6. *For $\Re(a) > 0$ we have*

$$\Psi(a) = \frac{1}{4} - \gamma B_1(a) - a + \frac{B_2(a)\zeta(2)}{2} + \sum_{n=3}^{\infty} \frac{(-1)^n B_n^{(n-1)}(a) H_{n-2}}{n!(n-2)}$$

Consequently we have the slowly convergent series of positive terms

$$\sum_{n=3}^{\infty} \frac{(-1)^{n+1} b_n^{(2)} H_{n-2}}{n-2} = \frac{\pi^2}{72} + \frac{\log(2\pi)}{2} - \frac{\gamma}{2} - \frac{3}{4}.$$

We define an Arakawa-Kaneko "Stieltjes constant" $\gamma(k,a)$ to be the constant term in the Laurent expansion of $Z_k(s,a)$ at $s=1$, namely

$$\gamma(k,a) = \lim_{s\to 1}\left(Z_k(s,a) - \frac{1}{s-1}\right). \tag{5.13}$$

From the derivative-shift property (4.4) it is clear that

$$\gamma(k,a) = -\frac{\partial}{\partial a} L\Gamma(k,a) = \frac{\partial}{\partial s}\xi_k(s,a)\bigg|_{s=0}, \tag{5.14}$$

so that we may also regard $-\gamma(k,a)$ as an Arakawa-Kaneko "digamma function", satisfying

$$\frac{\partial^m}{\partial a^m}\gamma(k,a) = (-1)^m m! Z_k(m+1,a) \qquad \text{for} \qquad m \geqslant 1 \tag{5.15}$$

([21], eq. (2.17)).

Theorem 7. *For $\Re(a) > 0$, $\Re(k) > 0$, and $\Re(a) + \Re(k) > 1$, the constant $\gamma(k,a)$ is given by*

$$\gamma(k,a) = \gamma + \sum_{n=1}^{\infty}(\zeta(k+n)-1) + \sum_{n=1}^{\infty}\frac{(-1)^n B_n^{(n)}(a)}{n!}H(k,n-1).$$

For positive integers k this may be expanded in the form

$$\gamma(k,a) = \gamma + k - \sum_{j=2}^{k}\zeta(j) - B_1(a)\zeta(k+1) + \sum_{n=2}^{\infty}\frac{(-1)^n B_n^{(n)}(a)}{n!}H(k,n-1)$$

for $\Re(a) > 0$.

Proof. By (5.14) this may be obtained by differentiating the result of Theorem 5 with respect to a using (2.3). $\square$

Again the $k=1$ case of this theorem gives a series expansion for $\psi(a)$ involving Bernoulli polynomials which is different from that given in [23]:

Corollary 8. *For $\Re(a) > 0$ we have*

$$\psi(a) - \psi(1) = B_1(a)\zeta(2) - 1 - \sum_{n=2}^{\infty}\frac{(-1)^n B_n^{(n)}(a) H_{n-1}}{n!(n-1)}.$$

Consequently we have the slowly convergent series of positive terms

$$\sum_{n=2}^{\infty}\frac{(-1)^{n+1} b_n H_{n-1}}{n-1} = 1 - \frac{\zeta(2)}{2}.$$

6. The values at integers $s > 1$

The values of Arakawa-Kaneko zeta functions at $s = 2$ have a simple expression in terms of the series $H(k, z)$:

Theorem 9. *For $\Re(a) > 0$, $\Re(k) > 0$, and $\Re(a) + \Re(k) > 1$ we have*

$$Z_k(2, a) = \zeta(k+1) + \sum_{n=1}^{\infty} (-1)^n \binom{a-1}{n} H(k, n).$$

For positive integers k this may be expanded in the form

$$Z_k(2, a) = \zeta(k+1) + \sum_{n=1}^{\infty} \frac{(-1)^n \binom{a-1}{n} H_n}{n^k} + \sum_{j=2}^{k} (-1)^j \zeta(j) \sum_{n=1}^{\infty} \frac{(-1)^n \binom{a-1}{n}}{n^{k+1-j}}$$

for $\Re(a) > 0$.

Proof. We put $s = 2$ in Theorem 3 and use (3.6) to obtain the first statement. The expansion of the second statement follows from (4.12). □

We remark that the well-known identity $Z_k(2, 1) = \zeta(k+1)$ [3, 8] follows directly from this theorem. Dividing by $a - 1$ and letting $a \to 1$ using (4.4) and (4.10) yields

$$Z_k(3, 1) = \frac{1}{2} \sum_{n=1}^{\infty} \frac{H(k, n)}{n} = \frac{1}{2} \sum_{m=1}^{\infty} \frac{H_m}{m^{k+1}} \tag{6.1}$$

which was given in [8] in the case of positive integers k. Comparison with Theorem 1 then permits the identifications

$$D_r^k \zeta_r(2, 1)\Big|_{r=1} = k! Z_{k+1}(2, 1) \tag{6.2}$$

and

$$D_r^k \zeta_r(3, 1)\Big|_{r=1} = k! \left(\frac{1}{2} Z_{k+1}(3, 1) - Z_{k+2}(2, 1) \right) \tag{6.3}$$

for positive integers k. Taking $a = 1/2$ and $k \in \mathbb{Z}^+$ in this theorem yields

$$Z_k(2, \frac{1}{2}) = \zeta(k+1) + \sum_{n=1}^{\infty} \frac{\binom{2n}{n} H_n}{4^n n^k} + \sum_{j=2}^{k} (-1)^j \zeta(j) \sum_{n=1}^{\infty} \frac{\binom{2n}{n}}{4^n n^{k+1-j}} \tag{6.4}$$

and then taking $k = 1$ yields

$$\zeta(2) = \frac{1}{2} \sum_{n=1}^{\infty} \frac{\binom{2n}{n} H_n}{n 4^n}. \tag{6.5}$$

This series stands in unusual contrast to known series (cf. e.g. [2, 7]) for $\zeta(2)$ in which the summands have a factor of $\binom{2n}{n}$ in the denominator.

Theorem 10. *For $\Re(a) > 0$ and integers $m \geqslant 3$, $k \geqslant 1$ we have*

$$\begin{aligned} Z_k(m,a) &= \frac{1}{m-1}\sum_{j=1}^{k-1}\frac{(-1)^{j-1}\zeta(k+1-j)}{(j-1)!}D_r^{j-1}\zeta_r(m-1,a)\Big|_{r=1} \\ &\quad + \frac{(-1)^{k+1}}{(m-1)!}\sum_{n=0}^{\infty}\frac{(-1)^n B_n^{(n+m-1)}(a)}{n!(n+m-2)^k}H_{n+m-2}. \end{aligned}$$

Equivalently we have

$$\begin{aligned} Z_k(m,a) &= \frac{1}{(m-1)!}\sum_{j=1}^{k-1}(-1)^{j-1}\zeta(k+1-j)D_r^{m-2}\zeta_r(j,0)\Big|_{r=1-a} \\ &\quad + \frac{(-1)^{k+1}}{(m-1)!}\sum_{n=0}^{\infty}\frac{(-1)^n B_n^{(n+m-1)}(a)}{n!(n+m-2)^k}H_{n+m-2}. \end{aligned}$$

Proof. Under the stated hypotheses, we substitute expression (4.12) for $H(k,n)$ into Theorem 3 to obtain

$$\begin{aligned} Z_k(m,a) &= \frac{(-1)^k}{m-1}\sum_{j=2}^{k}\frac{(-1)^j\zeta(j)}{(k-j)!}D_r^{k-j}\zeta_r(m-1,a)\Big|_{r=1} \\ &\quad + \frac{(-1)^{k+1}}{(m-1)!}\sum_{n=0}^{\infty}\frac{(-1)^n B_n^{(n+m-1)}(a)}{n!(n+m-2)^k}H_{n+m-2} \end{aligned} \tag{6.6}$$

using the second expression for $D_r^j\zeta_r(s,a)$ from Theorem 1. Rewriting the finite sum by replacing j with $k+1-j$ gives the first statement. Using the second expression for $D_r^j\zeta_r(s,a)$ from Theorem 1 gives the second statement. □

We remark that, by means of (2.15) and (2.17), the expressions involving $D_r^j\zeta_r(m-1,a)|_{r=1}$ occurring here may always be expressed as harmonic number series; furthermore the terms involving $B_n^{(n+m-1)}(a)$ may similarly be so expressed when $a=1$; this theorem therefore expresses $Z_k(m,1)$ in terms of Euler sums of weight at most $k+m-1$. This theorem gives different expressions of $Z_k(m,1)$ in terms of Euler sums than the expressions in [8] and different extensions to the case $a \neq 1$.

As an example, when $m=3$ and $a=1$, comparing (6.6) with (6.1) gives

$$\sum_{j=2}^{k}\frac{(-1)^j\zeta(j)}{(k-j)!}D_r^{k-j}\zeta_r(2,1)\Big|_{r=1} = \begin{cases} 4Z_k(3,1), & \text{if } k \text{ is even,} \\ 0, & \text{if } k \text{ is odd.} \end{cases} \tag{6.7}$$

This relation was given in ([3], Corollary 10(ii)), as may be seen from (6.2); the above theorem generalizes this kind of relation to general a and m. When

$k=1$ the sum over j in this theorem is empty and we obtain generalizations to general a of evaluations of zeta values as combinations of Euler sums, such as

$$\zeta(4) = \frac{1}{3}\sum_{n=1}^{\infty}\frac{H_n H_{n+1}}{(n+1)^2}, \tag{6.8}$$

$$\zeta(5) = \frac{1}{8}\sum_{n=1}^{\infty}\frac{(H_n^2 - H_n^{(2)})H_{n+1}}{(n+1)^2}, \tag{6.9}$$

$$\zeta(6) = \frac{1}{30}\sum_{n=1}^{\infty}\frac{(H_n^3 - 3H_n^{(2)}H_n + 2H_n^{(3)})H_{n+1}}{(n+1)^2}. \tag{6.10}$$

7. Logarithmic sums

Although the symmetric series identity of Corollary 2 does not hold for $j=0$ or $k=0$, we can derive an analogous identity for the antiderivative of $\zeta_r(s,a)$ with respect to its order r; this implies identities for series involving logarithms and harmonic numbers.

Theorem 11. *For $\Re(a) > 0$ and $\Re(s) > \Re(t)+1$ we have*

$$\begin{aligned}\int_t^{t+1}\zeta_r(s,a)\,dr &= \sum_{m=0}^{\infty}\frac{(-1)^{m+1}B_m^{(m)}(-t)}{m!(m+a)^s}\\ &= \frac{1}{\Gamma(s)}\sum_{n=0}^{\infty}\frac{(-1)^n B_n^{(n+s)}(a+1)}{n!}\log(n+s-t-1).\end{aligned}$$

Proof. Beginning with the $j=0$ case of Theorem 1, we integrate term by term using (2.3), invoking the difference equation (2.2) to consolidate terms in both expressions. □

Taking $a=1$, $t=-1$, and $s=1,2,3$ yields the examples

$$\sum_{m=0}^{\infty}\frac{(-1)^m b_m}{m+1} = \log 2; \tag{7.1}$$

$$\sum_{m=0}^{\infty}\frac{(-1)^m b_m}{(m+1)^2} = \sum_{n=1}^{\infty}\frac{\log(n+2)}{n(n+1)} - \log 2; \tag{7.2}$$

$$\sum_{m=0}^{\infty}\frac{(-1)^m b_m}{(m+1)^3} = \sum_{n=2}^{\infty}\frac{(H_n-1)\log(n+3)}{(n+1)(n+2)} - \frac{\log 3}{2}. \tag{7.3}$$

The first of these is well-known [13, 23]; the others may be compared to Example 9 of [5]. Taking $s=1$, $t=-1$, and $a=1/2$ yields

$$\sum_{m=0}^{\infty}\frac{(-1)^m b_m}{2m+1} = \sum_{n=0}^{\infty}\frac{C_n\log(n+2)}{4^{n+1}} \tag{7.4}$$

where $C_n = \binom{2n}{n}/(n+1)$ denotes the n-th Catalan number.

8. Other generalizations

If one interchanges the summations over index n and over index m in the series expansion of Theorem 3 one obtains

$$\Gamma(s)Z_k(s,a) = \sum_{m=1}^{\infty} \frac{1}{m^k} \sum_{n=0}^{\infty} \frac{(-1)^n B_n^{(n+s-1)}(a)}{n!(n+m+s-2)}. \tag{8.1}$$

From the estimate (2.10) and Lemma 4 we see that this rearranged series also converges absolutely and uniformly for s in compact subsets of $\mathbb{C} \setminus \{1, 0, -1, ...\}$ under the same hypotheses that $\Re(a) > 0$, $\Re(k) > 0$, and $\Re(a) + \Re(k) > 1$. By means of (3.4) this may be expressed in the form

$$Z_k(s,a) = \frac{1}{s-1} \sum_{m=1}^{\infty} \frac{\zeta_{1-m}(s-1,a)}{m^k} \tag{8.2}$$

giving a representation of the Arakawa-Kaneko zeta functions as Dirichlet series generated by integer-order Barnes zeta functions, convergent for all $s \in \mathbb{C}$. The special case $s = 2$, $a = 1$ of (8.2) is the identity $Z_k(2,1) = \zeta(k+1)$, and the case $s = 3$, $a = 1$ is precisely equation (6.1); therefore (8.2) may be viewed as a generalization to arbitrary s and a of these Euler sum evaluations. For integer values of $s > 1$ we may use the symmetry relation of Corollary 2 to express (8.2) as

$$\Gamma(s)Z_k(s,a) = \sum_{m=1}^{\infty} \frac{D_r^{s-2} \zeta_r(1,m)|_{r=1-a}}{m^k}. \tag{8.3}$$

Candelpergher and Coppo [5] also defined a modified zeta function $F_k(s)$ by the Mellin transform

$$F_k(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} \frac{e^{-t}}{1-e^{-t}} f_k(1-e^{-t}) dt \tag{8.4}$$

for $\Re(s) > 1$, where

$$f_k(z) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{b_n z^n}{n^k}, \tag{8.5}$$

and observed that $F_0(s) = \zeta(s) - \frac{1}{s-1}$. Analysis similar to Theorem 3 shows that

$$\Gamma(s)F_k(s) = \sum_{n=0}^{\infty} \frac{(-1)^n B_n^{(n+s)}(1)}{n!} \sum_{m=1}^{\infty} \frac{(-1)^{m+1} b_m}{m^k(m+n+s-1)} \tag{8.6}$$

for $s \in \mathbb{C}$ and $\Re(k) > -1$, providing explicit analytic continuation of $F_k(s)$ to the entire complex plane. Interchanging the summations allows the expression

$$F_k(s) = \sum_{m=1}^{\infty} \frac{(-1)^{m+1} b_m}{m^k} \zeta_{1-m}(s,1) \tag{8.7}$$

of $F_k(s)$ as a Dirichlet series generated by Bernoulli numbers of the second kind and integer-order Barnes zeta functions. The values of the modified zeta function $F_k(s)$ at positive integers are illustrated in Example 8 of [5]; equation (8.7) gives a general form of such evaluations. From (8.7) the values of $F_k(s)$ at the negative integers are given by

$$\begin{aligned} F_k(-n) &= n! \sum_{m=1}^{n+1} \frac{b_m B_{n+1-m}^{(1-m)}(1)}{m^k (n+1-m)!} \\ &= \sum_{m=0}^{n} \frac{m! b_{m+1} S(n, m \,|\, 1)}{(m+1)^k} \end{aligned} \tag{8.8}$$

in direct analogy to (4.13), (4.15); in particular for $k = 0$ we obtain the identity

$$\frac{B_{n+1}(1)}{n+1} = \frac{1}{n+1} - \sum_{m=0}^{n} m! b_{m+1} S(n, m \,|\, 1). \tag{8.9}$$

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