Bernoulli numbers and generalized factorial sums

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Abstract

We prove a pair of identities expressing Bernoulli numbers and Bernoulli numbers of the second kind as sums of generalized falling factorials. These are derived from an expression for the Mahler coefficients of degenerate Bernoulli numbers. As corollaries several unusual identities and congruences are derived.

1 Introduction

The *generalized falling factorial* $(x|\lambda)_n$ *with increment* λ is defined for positive integers n by

$$(x|\lambda)_n = \prod_{i=0}^{n-1}(x - i\lambda) \tag{1.1}$$

with the convention $(x|\lambda)_0 = 1$; clearly $(n|1)_n = (1|-1)_n = n!$ is the usual factorial. In this note we establish a pair of identities

$$B_n = \sum_{j=1}^{n+1} \frac{(-1)^{j-1}}{j} \binom{n+1}{j} \sum_{i=0}^{j-1} (i|j)_n, \tag{1.2}$$

$$(n+1)!\, n!\, b_n = \sum_{j=1}^{n+1} (-1)^{n+1-j} \binom{n+1}{j} \sum_{i=0}^{j-1} (i|j)_n, \tag{1.3}$$

for the *Bernoulli numbers* B_n and *Bernoulli numbers of the second kind* b_n. As consequences we deduce congruences such as

$$12 \cdot \sum_{j=2}^{p^r+3} \frac{(-1)^j}{j!\,(p^r+2)!\,(p^r+3-j)!} \sum_{i=1}^{j-1} (i|j)_{p^r+2} \equiv p^{2r} \pmod{p^{2r+1}\mathbb{Z}_p} \quad (1.4)$$

for odd primes p, where $\mathbb{Z}_p$ denotes the ring of p-adic integers. We also prove that if $n > 1$ is odd and d is any divisor of $n-2$, then

$$\sum_{m=2}^{n+1} \sum_{j=2}^{m} \sum_{i=1}^{j-1} \frac{(-1)^{m-j}\,(1|d)_m\,(i|j)_n}{d^m (m-j)!\,j!} = 0. \quad (1.5)$$

These well-known sequences of rational numbers are defined [11] by generating functions

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}, \quad (1.6)$$

$$\frac{t}{\log(1+t)} = \sum_{n=0}^{\infty} b_n t^n. \quad (1.7)$$

The numbers $n! b_n$ have also been called *Cauchy numbers* in [7]. We will deduce these identities from their relation to the *degenerate Bernoulli numbers* $\beta_n(\lambda)$ which are defined [6, 11] for $\lambda \neq 0$ by means of the generating function

$$\frac{t}{(1+\lambda t)^\mu - 1} = \sum_{n=0}^{\infty} \beta_n(\lambda) \frac{t^n}{n!} \quad (1.8)$$

where $\lambda\mu = 1$. These are polynomials in λ with rational coefficients; since $(1+\lambda t)^\mu \to e^t$ as $\lambda \to 0$ it is evident that $\beta_n(0) = B_n$; since $((1+t)^\mu - 1)/\mu \to \log(1+t)$ as $\mu \to 0$ we have $\lim_{\lambda\to\infty} \lambda^{-n}\beta_n(\lambda) = n!\,b_n$. Thus $\beta_n(\lambda)$ is a polynomial of degree n in λ whose constant term is B_n and whose leading coefficient is $n!\,b_n$. The numbers b_n and $\beta_n(\lambda)$ have been interpreted as divided differences of binomial coefficients [1] and are related to game theory [4].

2 Proof of Identities

The identities (1.2), (1.3) are deduced from the observation ([11], eq.(3.15)) that

$$a\beta_n(a) = \sigma_n(a, a-1) \quad (2.1)$$

for positive integers a, where

$$\sigma_n(\lambda, c) = \sum_{i=0}^{c} (i|\lambda)_n \tag{2.2}$$

is a generalized falling factorial sum. This equation may be given the following interpretation: for integers $i, n \geq 0$ one may consider that the generalized falling factorial $(i|a)_n$ is the product of all elements in the coset $i + (a)$ of the ideal (a) in the factor ring $\mathbb{Z}/an\mathbb{Z}$, a product which is well-defined modulo $an\mathbb{Z}$. Therefore, the integer $a\beta_n(a)$ is, up to a multiple of an, equal to the sum of all coset products over cosets of (a) in $\mathbb{Z}/an\mathbb{Z}$. It follows that the rational number $\beta_n(a)$ is, up to an integer multiple of n, the average coset product of (a) in $\mathbb{Z}/an\mathbb{Z}$.

If $f \in \mathbb{Z}[x]$ is a polynomial of degree N then it may be expressed in the form

$$f(x) = \sum_{m=0}^{N} a_m \binom{x}{m}, \tag{2.3}$$

where the *Mahler coefficients* $a_m \in \mathbb{Z}$ are uniquely determined by

$$a_m = \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} f(j) \tag{2.4}$$

(cf. [9], §52). Since the function $\lambda \mapsto \lambda\beta_n(\lambda)$ is a polynomial of degree $n+1$, we have

$$\lambda\beta_n(\lambda) = \sum_{m=1}^{n+1} \sum_{j=1}^{m} (-1)^{m-j} \binom{m}{j} \sigma_n(j, j-1) \binom{\lambda}{m} \tag{2.5}$$

as an identity of polynomials in $\mathbb{Z}[\lambda]$, and it similarly follows that

$$\sum_{j=1}^{m} (-1)^{m-j} \binom{m}{j} \sum_{i=0}^{j-1} (i|j)_n = 0 \qquad \text{when} \qquad m > n+1. \tag{2.6}$$

Theorem 1. *For all nonnegative integers n we have*

$$B_n = \sum_{j=1}^{n+1} \frac{(-1)^{j-1}}{j} \binom{n+1}{j} \sum_{i=0}^{j-1} (i|j)_n$$

and

$$(n+1)!\,n!\,b_n = \sum_{j=1}^{n+1}(-1)^{n+1-j}\binom{n+1}{j}\sum_{i=0}^{j-1}(i|j)_n.$$

Proof. From (2.5) we have

$$\lambda\beta_n(\lambda) = \sum_{m=1}^{n+1} a_m\binom{\lambda}{m} \tag{2.7}$$

where

$$a_m = \sum_{j=1}^{m}(-1)^{m-j}\binom{m}{j}\sigma_n(j,j-1) \tag{2.8}$$

as an identity of polynomials in λ. Equating coefficients of λ^{n+1} in (2.7) gives

$$n!\,b_n = a_{n+1}/(n+1)!, \tag{2.9}$$

giving the second statement.

For the first statement, we divide both sides of (2.7) by λ to obtain

$$\beta_n(\lambda) = \sum_{m=1}^{n+1}\frac{a_m}{m}\binom{\lambda-1}{m-1} \tag{2.10}$$

with a_m as in (2.8). Since $\binom{-1}{m-1} = (-1)^{m-1}$, evaluating (2.10) at $\lambda = 0$ then yields

$$\begin{aligned} B_n &= \sum_{m=1}^{n+1}\frac{(-1)^{m-1}a_m}{m} \\ &= \sum_{m=1}^{n+1}\sum_{j=1}^{m}\frac{(-1)^{j-1}}{m}\binom{m}{j}\sum_{i=0}^{j-1}(i|j)_n \\ &= \sum_{m=1}^{n+1}\sum_{j=1}^{m}\frac{(-1)^{j-1}}{j}\binom{m-1}{j-1}\sum_{i=0}^{j-1}(i|j)_n. \end{aligned} \tag{2.11}$$

From the familiar property $\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$ we have

$$\sum_{m=1}^{n+1}\binom{m-1}{j-1} = \binom{n+1}{j}, \tag{2.12}$$

and therefore the triple sum of (2.11) becomes the double sum of the first statement of the theorem. □

Remark. In these identities the sums over index i run from 0 to $j-1$ so as to make them valid for $n=0$; when $n>0$ the $i=0$ term contributes nothing, so the inner sum may be indexed from $i=1$ to $j-1$, and in turn the outer sum may be indexed from $j=2$ to $n+1$ when $n>0$.

3 Bernoulli numbers

The above theorem expresses the Bernoulli number B_n as a sum of integer multiples of $1/j$ for $1 \le j \le n+1$. As such it may be compared to identities such as

$$B_n = \sum_{j=0}^{n} \frac{1}{j+1} \sum_{i=0}^{j} \binom{j}{i} i^n \tag{3.1}$$

and

$$B_n = \sum_{j=1}^{n+1} \frac{(-1)^{j-1}}{j} \binom{n+1}{j} \sum_{i=1}^{j} i^n \tag{3.2}$$

(cf. [3]), although it gives a different such expression. The following result may also be deduced from the above theorem and well-known fact that $B_{2k+1}=0$ for positive integers k.

Corollary 1. *If $n>1$ is odd then*

$$\sum_{j=2}^{n+1} \frac{(-1)^{j-1}}{j} \binom{n+1}{j} \sum_{i=1}^{j-1} (i|j)_n = 0.$$

We may also derive the following congruences for the Bernoulli numbers. These are generalizations of the well-known fact that $pB_{p-1} \equiv -1 \pmod{p}$ for odd primes p.

Corollary 2. *If p is an odd prime and $\lfloor n/p \rfloor = k$, then*

$$pB_n \equiv \sum_{j=1}^{k+1} \frac{(-1)^{j-1}}{j} \binom{n+1}{jp} \sum_{i=1}^{jp-1} (i|jp)_n \pmod{p^{k+1}\mathbb{Z}_p}.$$

For the case $n \equiv -1 \pmod{p}$ the congruence may be improved to

$$pB_{kp-1} \equiv \sum_{j=1}^{k} \frac{(-1)^{j-1}}{j} \binom{kp}{jp} \sum_{i=1}^{jp-1} (i|jp)_{kp-1} \pmod{kp^{k+1}\mathbb{Z}_p},$$

and for $p > 3$ and $k > 1$ we also have

$$pB_{kp-1} \equiv \sum_{j=1}^{k} \frac{(-1)^{j-1}}{j} \binom{k}{j} \sum_{i=1}^{jp-1} (i|jp)_{kp-1} \pmod{kp^3\mathbb{Z}_p}.$$

Proof. If j is not a multiple of p then each falling factorial $(i|j)_n$ with $1 \leq i < j$ lies in $p^k\mathbb{Z}$, so the first statement follows directly from the theorem. When $n = kp - 1$ and j is not a multiple of p the binomial coefficient $\binom{kp}{j}$ lies in $kp\mathbb{Z}_p$, while each falling factorial $(i|j)_{kp-1}$ with $1 \leq i < j$ lies in $p^{k-1}\mathbb{Z}$. Therefore,

$$pB_{kp-1} \equiv \sum_{j=1}^{k} \frac{(-1)^{j-1}}{j} \binom{kp}{jp} \sum_{i=1}^{jp-1} (i|jp)_{kp-1} \pmod{kp^{k+1}\mathbb{Z}_p}. \tag{3.3}$$

Invoking the Kazandzidis supercongruence

$$\binom{kp}{jp} \equiv \binom{k}{j} \pmod{jk(k-j)p^3\mathbb{Z}_p} \tag{3.4}$$

(cf. [8]) for $p > 3$ gives the second result. □

4 Bernoulli numbers of the second kind

The following corollary expresses the well-known fact [5, 12] that $b_n \neq 0$ for all n.

Corollary 3. *For all integers $n \geq 0$,*

$$\sum_{j=1}^{n+1} (-1)^j \binom{n+1}{j} \sum_{i=0}^{j-1} (i|j)_n \neq 0.$$

The rational numbers b_n have recently been expressed as p-adically convergent sums of traces of algebraic integers [12, 13]. Here we present a congruence result which illustrates this description.

Corollary 4. *For all integers $n \geq 0$ and all primes p,*

$$p^{\lfloor n/(p-1) \rfloor} \sum_{j=1}^{n+1} \frac{(-1)^{n+1-j}}{j!\, n!\, (n+1-j)!} \sum_{i=0}^{j-1} (i|j)_n \quad \textit{is} \quad p\textit{-integral}$$

and

$$\sum_{j=1}^{n+1} \frac{(-1)^{n+1-j}}{j!\, n!\, (n+1-j)!} \sum_{i=0}^{j-1} (i|j)_n$$
$$\equiv -\mathrm{Tr}(\zeta_p(\zeta_p - 1)^{-n}) \pmod{p^{1-\lfloor (n+p-1)/(p^2-p) \rfloor}\mathbb{Z}_p},$$

where ζ_p *is any primitive* p*-th root of unity and* Tr *denotes the trace map from* $\mathbb{Q}(\zeta_p)$ *to* $\mathbb{Q}$.

Proof. By our theorem, the left member of the above congruence is simply b_n. For odd primes p these statements follow from [13], Corollary 1. For $p = 2$ the first statement follows from [2], Theorem 2 and the second follows from [12], Theorem 1. □

Remarks. It is well-known and easy to see that $p^{\lfloor n/(p-1) \rfloor}/n!$ is always p-integral, but there seems to be no obvious reason why the given expression is p-integral. In the above congruence, both members are rational numbers which lie in $p^{-\lfloor n/(p-1) \rfloor}\mathbb{Z}_p$ generically and the congruence says that they agree to approximately $1+((n-1)/p)$ digits p-adically. This congruence is unusual in that the left member, which is independent of the prime p, is simultaneously congruent to the right member for all primes p.

We conclude this section with another congruence illustrating the unusual arithmetic properties of the sequence $\{b_n\}$; the left member ostensibly should have a factor of p^{2r} in its denominator, rather than its numerator.

Corollary 5. *For all odd primes* p *and all* $r > 0$,

$$12 \cdot \sum_{j=2}^{p^r+3} \frac{(-1)^j}{j!\, (p^r+2)!\, (p^r+3-j)!} \sum_{i=1}^{j-1} (i|j)_{p^r+2} \equiv p^{2r} \pmod{p^{2r+1}\mathbb{Z}_p}.$$

Proof. This is a restatement of the congruence $12 b_{p^r+2} \equiv p^{2r} \pmod{p^{2r+1}\mathbb{Z}_p}$ given in [11], Theorem 4.4. □

5 Other generalizations of Bernoulli numbers

The *Bernoulli numbers* $B_n^{(w)}$ *of order* w are defined by the generating function

$$\left(\frac{t}{e^t - 1}\right)^w = \sum_{n=0}^{\infty} B_n^{(w)} \frac{t^n}{n!}, \tag{5.1}$$

and the numbers $B_n^{(n)}$, where the order equals the degree, are called *Nörlund numbers* [5, 12]. Comparing the theorem with the identity [5]

$$\frac{B_n^{(n)}}{n!} = \sum_{k=0}^{n} (-1)^{n-k} b_k \tag{5.2}$$

yields the triple sum identity

$$\frac{B_n^{(n)}}{n!} = \sum_{k=0}^{n} \sum_{j=1}^{k+1} \frac{(-1)^{n+1-j}}{j!\, k!\, (k+1-j)!} \sum_{i=0}^{j-1} (i|j)_k \tag{5.3}$$

for the Nörlund numbers.

The next corollary gives a regular Kummer-type congruence for the degenerate Bernoulli numbers. It applies to β_m rather than to the divided sequence β_m/m, and also differs from other such congruences for Bernoulli numbers in that the case $p-1|m$ is not excluded (cf. e.g. [9], §61).

Corollary 6. *Suppose p is a prime number and $\lambda \in p^k\mathbb{Z}_p$ with $k > 0$. If p is odd let $m \equiv n \pmod{(p-1)p^e}$ and if $p = 2$ let $m \equiv n \pmod{2^{e+1}}$. Then*

$$\beta_m(\lambda) \equiv \beta_n(\lambda) \qquad (\operatorname{mod} p^B\mathbb{Z}_p)$$

where $B = \min\{m, n, e+1\} - k$.

Proof. The generating function

$$\frac{(1+\lambda t)^{(c+1)\mu} - 1}{(1+\lambda t)^{\mu} - 1} = \sum_{m=0}^{\infty} \sigma_m(\lambda, c) \frac{t^m}{m!}; \tag{5.4}$$

[11], eq. (2.3) reveals that $\{\sigma_m(\lambda, c)\}$ is the *degenerate number sequence* associated to the polynomial $h(T) = (T^{c+1} - 1)/(T - 1) \in \mathbb{Z}[T]$ in the terminology of [10]. Therefore by [10], Theorem 1.2 we have

$$\sigma_m(\lambda, c) \equiv \sigma_n(\lambda, c) \qquad (\operatorname{mod} p^A\mathbb{Z}_p) \tag{5.5}$$

under the stated hypotheses on m, n, and λ, where $A = \min\{m, n, e+1\}$. Taking λ to be a positive integer multiple of p and $c = \lambda - 1$ yields

$$\lambda\beta_m(\lambda) \equiv \lambda\beta_n(\lambda) \qquad (\operatorname{mod} p^A\mathbb{Z}_p) \tag{5.6}$$

via (2.1). Dividing by λ gives the stated congruence for positive integers λ; since the set of positive integer multiples of p is dense in $p\mathbb{Z}_p$, the result follows for all $\lambda \in p\mathbb{Z}_p$. □

We conclude with the following triple sum identity, for which we would like to find a combinatorial explanation.

Corollary 7. *If $n > 1$ is odd and d is any divisor of $n-2$, positive or negative, then*

$$\sum_{m=2}^{n+1}\sum_{j=2}^{m}\sum_{i=1}^{j-1}\frac{(-1)^{m-j}\,(1|d)_m\,(i|j)_n}{d^m(m-j)!\,j!}=0.$$

Proof. We recently proved [11], Theorem 4.7 that if $n > 1$ is odd and d is any divisor of $n-2$, then $\beta_n(1/d) = 0$. From (2.5) we have

$$\begin{aligned}
0 &= \beta_n(1/d)/d \\
&= \sum_{m=1}^{n+1}\binom{1/d}{m}\sum_{j=1}^{m}(-1)^{m-j}\binom{m}{j}\sum_{i=1}^{j-1}(i|j)_n \\
&= \sum_{m=1}^{n+1}\frac{(1/d|1)_m}{m!}\sum_{j=1}^{m}(-1)^{m-j}\binom{m}{j}\sum_{i=1}^{j-1}(i|j)_n \\
&= \sum_{m=1}^{n+1}\frac{(1|d)_m}{d^m m!}\sum_{j=1}^{m}(-1)^{m-j}\binom{m}{j}\sum_{i=1}^{j-1}(i|j)_n \qquad (5.7)\\
&= \sum_{m=1}^{n+1}\sum_{j=1}^{m}\sum_{i=1}^{j-1}\frac{(-1)^{m-j}\,(1|d)_m\,(i|j)_n}{d^m(m-j)!\,j!}.
\end{aligned}$$

The terms corresponding to $j = 1$ or to $m = 1$ contribute nothing, so the corollary is proved. □

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