On the binary expansion of the odd Catalan numbers

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Abstract

Let $c_n = \frac{1}{n+1}\binom{2n}{n}$ be the nth Catalan number. In this paper, we look at some of the arithmetic properties of c_n. For example, we show that $w_2(c_n) \gg \log\log n$ for all $n \geqslant 3$ provided that c_n is odd, where $w_2(m)$ is the Hamming weight (or the binary sum of digits) of the positive integer m. We also determine all instances in which c_n is a base 2 palindrome, and prove that the first k odd Catalan numbers are always distinct modulo 2^{k+1}.

1 Introduction

Let

$$b_n = \binom{2n}{n} \qquad \text{and} \qquad c_n = \frac{1}{n+1}\binom{2n}{n}$$

be the nth middle binomial coefficient and the nth Catalan number, respectively. For positive integers m and $g \geqslant 2$ let $w_g(m)$ be the sum of the base g-digits of m. In [4], it was shown that both inequalities

$$w_g(b_n) \gg \varepsilon(n)(\log n)^{1/2} \qquad \text{and} \qquad w_g(c_n) \gg \varepsilon(n)(\log n)^{1/2}$$

hold for all n in a set of asymptotic density 1, where $\varepsilon(x)$ is any function such that $\lim_{x\to\infty} \varepsilon(x) = 0$. In this note, we revisit this problem when $g = 2$ and c_n is odd. Our initial motivation was to find all positive integers n such that c_n is a binary palindrome. Recall that if the g-ary expansion of a positive integer m is

$$m = m_0 + m_1 g + \cdots + m_k g^k, \qquad \text{where} \quad w_i \in \{0, 1, \ldots, g-1\}, \quad w_k \neq 0,$$

then m is called a *base g palindrome* if $w_i = w_{k-i}$ holds for all $i = 0, \ldots, k$. When $g = 2$, a base 2 palindrome is also called a *binary palindrome*. Since binary palindromes start with the binary digit 1, they must also end with the binary digit 1, and, in particular, they must be odd. Palindromes in other sequences have been studied previously, one such example being the paper [5] in which all positive integers n such that $10^n - 1$ is a binary palindrome have been determined.

Our results are the following.

Theorem 1. *If c_n is a binary palindrome, then $n = 1,\ 3$.*

Theorem 2. *Both inequalities*

$$w_2(b_n) \gg \log\log n \qquad \text{and} \qquad w_2(c_n) \gg \log\log n$$

hold for all $n \geqslant 3$ such that c_n is odd.

Theorem 3. *For all $k > 0$, the first k odd Catalan numbers are distinct modulo 2^{k+1}.*

2 Proofs

It is well-known that c_n is odd if and only if $n = 2^k - 1$ for some $k \geqslant 1$. We start with a useful auxiliary result.

Lemma 4. *Assume that $n = 2^k - 1$ with $k \geqslant 4$ and put $\delta \in \{0, 1\}$ such that $k \equiv \delta \pmod 2$. Then the following estimate holds*

$$c_n = \frac{2^{2n-\lfloor 3k/2 \rfloor}}{2^{\delta/2}\pi^{1/2}}(1+\zeta_n), \qquad \textit{where} \qquad \frac{1}{2^{k+2}} < \zeta_n < \frac{1}{2^{k+1}}.$$

Proof. We start with the Stirling's formula

$$n! = (n/e)^n\sqrt{2\pi n}e^{\theta_n}, \qquad \text{where} \qquad \frac{1}{12(n+1)} < \theta_n < \frac{1}{12n}.$$

Thus,

$$c_n = \frac{1}{n+1}\binom{2n}{n} = \frac{2^{2n}}{\pi^{1/2}(n+1)^{3/2}}e^{\theta_{2n}-2\theta_n+\lambda_n},$$

where

$$\lambda_n = \frac{1}{2}\log\left(1+\frac{1}{n}\right) \in \left(\frac{1}{2(n+1)}, \frac{1}{2n}\right).$$

Now clearly

$$\begin{aligned} \theta_{2n} - 2\theta_n + \lambda_n &< \frac{1}{24n} - \frac{1}{6(n+1)} + \frac{1}{2n} = \frac{9n+13}{24n(n+1)}, \\ \theta_{2n} - 2\theta_n + \lambda_n &> \frac{1}{24(n+1)} - \frac{1}{6n} + \frac{1}{2(n+1)} = \frac{9n-4}{24n(n+1)}. \end{aligned}$$

Now since

$$1 + x < e^x < 1 + x + x^2 \qquad (x < 1/2),$$

we get that

$$e^{\theta_{2n}-2\theta_n-\lambda_n} = 1 + \zeta_n,$$

where

$$\begin{aligned} \zeta_n &> \frac{9n-4}{24n(n+1)} \qquad \text{and} \\ \zeta_n &< \frac{9n+13}{24n(n+1)} + \frac{(9n+13)^2}{24^2n^2(n+1)^2} < \frac{9n+17}{24n(n+1)}, \end{aligned}$$

where the last inequality holds for all $n \geqslant 11$. Thus,

$$\frac{9n-4}{24n(n+1)} < \zeta_n < \frac{9n+17}{24n(n+1)},$$

which leads easily to the conclusion that

$$\frac{1}{2^{k+2}} < \zeta_n < \frac{1}{2^{k+1}}$$

for $k \geqslant 4$, and thus completes the proof of the lemma. $\square$

Proof of Theorem 1. Assume that c_n is a binary palindrome. In particular, it must be odd, therefore $n = 2^k - 1$ for some positive integer k. Assume that $k \geqslant 10$. Lemma 4 shows that

$$c_n = \frac{2^N}{2^{\delta/2}\pi^{1/2}}(1+\zeta_k), \qquad 0 < \zeta_k < \frac{1}{2^{11}},$$

where $\delta \in \{0,1\}$ and $N = 2n - \lfloor 3k/2 \rfloor$. The first few digits of the binary expansions of $1/\sqrt{\pi}$ and $1/\sqrt{2\pi}$ are

$$\frac{1}{\sqrt{\pi}} = 0.1001000001\ldots \qquad \text{and} \qquad \frac{1}{\sqrt{2\pi}} = 0.01100110001\ldots,$$

respectively. Since ζ_n is positive but $< 1/2^{11}$, we see that the first significant four binary digits of c_n are either 1001 or 1100, according to whether k is even or odd, respectively. Should c_n be a binary palindrome, we would get that $c_n \equiv 9,\ 3 \pmod{16}$, but this is false because it is known that $c_n \equiv 5 \pmod 8$ (see [2]). $\square$

Proof of Theorem 2. Assume that c_n is odd. Thus, $n = 2^k - 1$ for some k. Assume that k is large. We put again $\delta \in \{0,1\}$ such that $k \equiv \delta \pmod 2$. Lemma 4 shows roughly that the first $k \asymp \log n$ digits of c_n are related to the first k digits of either $1/\sqrt{\pi}$ or $1/\sqrt{2\pi}$, according to whether k is even or odd, respectively. To be more precise, given k, let k_1 be the position of the right most digit of 0 among the first k binary digits of $1/\sqrt{2^\delta\pi}$. Then the first significant $k_1 - 1$ digits of c_n are the same as the first significant $k_1 - 1$ digits of $1/\sqrt{2^\delta\pi}$. Moreover, observe that if $n = 2^k - 1$, then $w_2(b_n) = w_2(c_n)$. Thus, Theorem 2 will follow immediately from the following result.

Lemma 5. *For $\delta \in \{0,1\}$, $d \in \{0,1\}$, $s \geqslant 1$, let $N(\delta,d,s)$ be the number of binary digits whose value is d among the first s significant binary digits of $1/\sqrt{2^\delta \pi}$. Then*

$$N(\delta,d,s) \gg \log s \qquad \textit{for} \qquad s \geqslant 4.$$

Proof. We shall prove the lemma only for $d=1$, since for $d=0$ the proof is analogous. Write

$$\frac{1}{\sqrt{2^\delta \pi}} = \frac{1}{2^{\alpha_1}} + \frac{1}{2^{\alpha_2}} + \ldots, \qquad \text{where} \qquad 1 \leqslant \alpha_1 < \alpha_2 < \cdots .$$

We show that $\alpha_{s+1} \ll \alpha_s$ holds for all $s \geqslant 1$, which certainly implies the conclusion of the lemma. Well, observe that

$$\left| \frac{1}{\sqrt{2^\delta \pi}} - \frac{A_s}{B_s} \right| \ll \frac{1}{2^{\alpha_{s+1}}}, \tag{1}$$

where

$$B_s = 2^{\alpha_s}, \qquad \text{and} \qquad \frac{A_s}{B_s} = \frac{1}{2^{\alpha_1}} + \cdots + \frac{1}{2^{\alpha_s}}.$$

Multiplying both sides of estimate (1) with

$$2^\delta \left(\frac{1}{\sqrt{2^\delta \pi}} + \frac{A_s}{B_s} \right) = O(1),$$

we get

$$\left| \frac{1}{\pi} - \frac{2^\delta A_s^2}{B_s^2} \right| \ll \frac{1}{2^{\alpha_{s+1}}}.$$

On the other hand, it is known that the left hand side above is $\gg B_s^{-2K}$ for some positive constant K, which can be taken to be 7.02 for large s by Hata's work [3]. Hence, we get $\alpha_{s+1} \leqslant 2K\alpha_s + O(1) \ll \alpha_s$, which concludes the proof of the lemma. $\square$

Proof of Theorem 3. We consider the 2-adic Morita gamma function Γ_2 defined (see [1], p. 368) for positive integers n by

$$\Gamma_2(n) = (-1)^n \prod_{\substack{0<j<n \\ 2 \nmid j}} j;$$

it extends uniquely to a continuous function from $\mathbb{Z}_2$ to $\mathbb{Z}_2^\times$, where $\mathbb{Z}_2$ denotes the ring of 2-adic integers, and satisfies the functional equation

$$\Gamma_2(x+1) = \begin{cases} -x\Gamma_2(x), & x \in \mathbb{Z}_2^\times, \\ -\Gamma_2(x), & x \in 2\mathbb{Z}_p. \end{cases}$$

From the relation (see [1], p. 368)

$$\Gamma_2(1+n) = (-1)^{n+1}\frac{n!}{2^{[n/2]}[n/2]!},$$

we obtain

$$\frac{c_{2^k-1}}{c_{2^{k-1}-1}} = \left(\frac{2^k-1}{2^{k+1}-1}\right)\frac{\Gamma_2(2^{k+1})}{\Gamma_2(2^k)^2}.$$

Now Γ_2 has a power series expansion

$$\Gamma_2(x) = \sum_{j=0}^{\infty} \gamma_j x^j \quad (x \in 4\mathbb{Z}_2)$$

(see [1], p. 379), with all γ_j lying in the field $\mathbb{Q}_2$ of 2-adic numbers, with $\gamma_0 = 1$, $\gamma_1 \in \mathbb{Z}_2^\times$, and $2^j\gamma_j \in \mathbb{Z}_2$ for $j \geqslant 2$. Therefore, we have

$$\Gamma_2(2^j) \equiv 1 + 2^j\gamma_1 \pmod{2^{2j-2}\mathbb{Z}_2}$$

for all positive integers j, and thus

$$\frac{\Gamma_2(2^{k+1})}{\Gamma_2(2^k)^2} \equiv 1 \pmod{2^{2k-2}\mathbb{Z}_2}$$

for all positive integers k. Since $(1-2^{k+1})^{-1} \equiv 1 + 2^{k+1} \pmod{2^{2k+2}\mathbb{Z}_2}$, we obtain the congruence

$$c_{2^k-1} \equiv (1+2^k)c_{2^{k-1}-1} \pmod{2^{2k-2}\mathbb{Z}}$$

for all positive integers k. This implies that c_{2^k-1} and $c_{2^{k-1}-1}$ are always congruent modulo 2^k but are always incongruent modulo 2^{k+1}.

The theorem now follows by induction on k. We have $c_1 = 1$ and $c_3 = 5$ which are certainly distinct modulo 8, so the case $k = 2$ is cleared. Now assume $c_1, \ldots, c_{2^{k-1}-1}$ are all distinct modulo 2^k. Since $c_{2^k-1} \equiv c_{2^{k-1}-1}$

(mod 2^k) we have $c_{2^k-1} \not\equiv c_{2^j-1} \pmod{2^k}$ for $1 \leqslant j < k-1$; but also by the above $c_{2^k} \not\equiv c_{2^{k-1}-1} \pmod{2^{k+1}}$, which shows that $c_1, \ldots, c_{2^k-1}$ are all distinct modulo 2^{k+1}. □

In conclusion, we remark that the 2-adic limit $\lim_{k\to\infty} c_{2^k-1}$ of odd Catalan numbers exists in $\mathbb{Z}_2$ and is equal to $\prod_{k\geqslant 1} \Gamma_2(2^k)^{-1}$. We leave the determination of the transcendence or algebraicity of this limit as an open problem to the reader.

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