A p-adic formula for the Nörlund numbers and for Bernoulli numbers of the second kind

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Abstract

We give formulas expressing the Nörlund numbers and the Bernoulli numbers of the second kind as p-adically convergent sums of traces of algebraic integers for odd primes p. We use these formulas to derive new congruences and divisibility results for these sequences, including analogues of Kummer's congruences.

Keywords: Nörlund numbers, Bernoulli numbers of second kind, Cauchy numbers, congruences, p-adic analysis

1. Introduction

The *Bernoulli numbers of order w*, $B_n^{(w)}$, are the rational numbers defined [8] by the generating function

$$\left(\frac{t}{e^t-1}\right)^w = \sum_{n=0}^{\infty} B_n^{(w)} \frac{t^n}{n!}. \tag{1.1}$$

For $n = w$ the numbers $B_n^{(n)}$ are called *Nörlund numbers* [4], or *Cauchy numbers of the second type* ([3], [7]), and may also be determined by the generating function

$$\frac{t}{(1+t)\log(1+t)} = \sum_{n=0}^{\infty} B_n^{(n)} \frac{t^n}{n!} \tag{1.2}$$

(cf. [8]). The first few values are $B_0^{(0)} = 1$, $B_1^{(1)} = -1/2$, $B_2^{(2)} = 5/6$, $B_3^{(3)} = -9/4$, $B_4^{(4)} = 251/30$, $B_5^{(5)} = -475/12$. One important role they play in combinatorial analysis is through the formula

$$B_n^{(n)} = \int_0^1 (x-1)(x-2)\cdots(x-n)\,dx \tag{1.3}$$

(cf. [8]). The *Bernoulli numbers of the second kind* b_n are the rational numbers determined ([2], [5]) by the generating function

$$\frac{t}{\log(1+t)} = \sum_{n=0}^{\infty} b_n t^n. \tag{1.4}$$

The numbers $n!b_n$ have also been called *Cauchy numbers of the first type* ([3], [7]), and may be defined by

$$n!b_n = \int_0^1 x(x-1)(x-2)\cdots(x-n+1)\,dx. \tag{1.5}$$

The first few values are $b_0 = 1$, $b_1 = 1/2$, $b_2 = -1/12$, $b_3 = 1/24$, $b_4 = -19/720$, $b_5 = 3/160$. These sequences are related by the formulas [4]

$$\frac{B_n^{(n)}}{n!} = \sum_{j=0}^{n}(-1)^{n-j}b_j \qquad \text{and} \qquad b_n = \frac{B_n^{(n)}}{n!} + \frac{B_{n-1}^{(n-1)}}{(n-1)!}. \tag{1.6}$$

The main results of this paper are the following expressions of these sequences as p-adically convergent sums of traces of certain algebraic integers:

Theorem 1. *Let p be an odd prime and for each $r \geq 0$ let ζ_r denote any primitive p^{r+1}-th root of unity. Then for all nonnegative integers n we have*

$$(-p)^n \frac{B_n^{(n)}}{n!} = -\sum_{r=0}^{\infty} \mathrm{Tr}_r\left(\left(\frac{p}{1-\zeta_r}\right)^n\right)$$

as a p-adically convergent sum of integers, where Tr_r denotes the trace map from $\mathbb{Q}(\zeta_r)$ to $\mathbb{Q}$, and for all positive integers n we have

$$(-p)^n b_n = -\sum_{r=0}^{\infty} \mathrm{Tr}_r\left(\zeta_r\left(\frac{p}{1-\zeta_r}\right)^n\right)$$

as a p-adically convergent sum of integers.

A similar formula for $p = 2$ was given in [9], where it was used to prove the conjectures of Adelberg [1] concerning the 2-adic digits of $B_n^{(n)}/n!$ and of b_n. In Section 3 we apply Theorem 1 to prove new congruences for these sequences, including a version of Kummer congruences.

2. Proof of p-adic formula.

Throughout this paper p will denote an odd prime, $\mathbb{Z}_p$ the ring of p-adic integers, $\mathbb{Q}_p$ the field of p-adic numbers, and ord_p the p-adic valuation normalized by $\mathrm{ord}_p p = 1$. Clearly $\zeta = \zeta_r$ is a primitive p^{r+1}-th root of unity if and only if $\zeta^{p^r} = \zeta_0$ is a primitive p-th root of unity, so the minimal polynomial for ζ_r over $\mathbb{Q}$ is the p^{r+1}-th cyclotomic polynomial $\Phi_{p^{r+1}}(t) = \Phi_p(t^{p^r})$, where $\Phi_p(t) = 1 + t + \cdots + t^{p-1}$ is the p-th cyclotomic polynomial. It is well known that $\mathbb{Q}(\zeta_r)$ is a cyclic extension of $\mathbb{Q}$ with Galois group $\mathrm{Gal}(\mathbb{Q}(\zeta_r)/\mathbb{Q}) = \{\sigma_j : j \in (\mathbb{Z}/p^{r+1}\mathbb{Z})^\times\} \cong (\mathbb{Z}/p^{r+1}\mathbb{Z})^\times$, where σ_j is the automorphism of $\mathbb{Q}(\zeta_r)$ induced by $\zeta_r \mapsto \zeta_r^j$. Since $1 - \zeta_0$ is a root of the p-Eisenstein polynomial $\Phi_p(1-t) = p - \binom{p}{2}t + \cdots + t^{p-1}$ we have $\mathrm{ord}_p(1-\zeta_0) = 1/(p-1)$, and thus $\mathrm{ord}_p(1-\zeta_r) = 1/(p^r(p-1))$ for all r; hence the extension $K_r = \mathbb{Q}_p(\zeta_r)$ of $\mathbb{Q}_p$ is totally ramified of degree $p^r(p-1)$, with ring of integers $\mathfrak{O}_r = \{x \in K_r : \mathrm{ord} x \geq 0\}$, maximal ideal $\mathfrak{P}_r = \{x \in K_r : \mathrm{ord} x \geq 1/((p^r(p-1))\}$, and residue class field $\mathfrak{O}_r/\mathfrak{P}_r$ isomorphic to $\mathbb{Z}/p\mathbb{Z}$ (cf. [6]).

Proof of Theorem 1. We begin with the partial fraction decomposition

$$\frac{p}{T^p - 1} - \frac{1}{T-1} = \sum_{\zeta=\zeta_0} \frac{\zeta}{T-\zeta}, \tag{2.1}$$

where the sum is over all primitive p-th roots of unity $\zeta = \zeta_0$. We substitute $T = (1-pt)^{p^r}$ and multiply by $-p^{r+1}t$ to obtain

$$\frac{-p^{r+2}t}{(1-pt)^{p^{r+1}} - 1} - \frac{-p^{r+1}t}{(1-pt)^{p^r} - 1} = -\sum_{\zeta=\zeta_0} \frac{p^{r+1}\zeta t}{(1-pt)^{p^r} - \zeta}. \tag{2.2}$$

Note that the sum on the right in (2.2) is a power series with integer coefficients, since this is certainly true on the left. If we sum this equation from $r = 0$ to $r = s$, the left side telescopes, yielding

$$\frac{-p^{s+2}t}{(1-pt)^{p^{s+1}} - 1} - 1 = -\sum_{r=0}^{s} \sum_{\zeta=\zeta_0} \frac{p^{r+1}\zeta t}{(1-pt)^{p^r} - \zeta}. \tag{2.3}$$

Since $((1-pt)^{p^r} - \zeta)/(1-\zeta)$ is a unit in the power series ring $\mathbb{Z}_p[\zeta][[t]]$, the r-th term in the sum indexed by r in (2.3) lies in $p^{r+1}\mathbb{Z}_p[[t]]$, hence the p-adic limit of partial sums as $s \to \infty$ exists. Since $((1+t)^a - 1)/a \to \log(1+t)$ as $a \to 0$, we have

$$\frac{-pt}{\log(1-pt)} - 1 = -\sum_{r=0}^{\infty} \sum_{\zeta=\zeta_0} \frac{p^{r+1}\zeta t}{(1-pt)^{p^r} - \zeta} \tag{2.4}$$

as an identity in $\mathbb{Z}[[t]]$. Expanding the left side of (2.4) as a power series gives

$$\sum_{n=0}^{\infty} (-p)^n b_n t^n = 1 - \sum_{r=0}^{\infty} \sum_{\zeta=\zeta_0} \frac{p^{r+1}\zeta t}{(1-pt)^{p^r} - \zeta}. \tag{2.5}$$

We define rational integers $c_{r,n}$ by

$$\sum_{n=0}^{\infty} c_{r,n} t^n = -\sum_{\zeta=\zeta_0} \frac{p^{r+1}\zeta t}{(1-pt)^{p^r} - \zeta} \tag{2.6}$$

so that $(-p)^n b_n = \sum_{r=0}^{\infty} c_{r,n}$ as a convergent sum in $\mathbb{Z}_p$ for all $n > 0$.

If $Q(t) = \prod_{i=1}^{n}(1-\alpha_i t)$ is a polynomial of degree n with distinct roots and $P(t)$ is a polynomial of degree less than n, it is easily seen that there is a partial fraction decomposition of $P(t)/Q(t)$

as $\sum_{i=1}^n a_i/(1-\alpha_i t)$ where $a_i = -\alpha_i P(\alpha_i^{-1})/Q'(\alpha_i^{-1})$ for all i. Thus since $(1-pt)^{p^r} - \zeta_0 = 0$ whenever $pt = 1-\zeta_r$ for any of the p^r primitive p^{r+1}-th roots of unity ζ_r satisfying $\zeta_r^{p^r} = \zeta_0$, we have by partial fraction decomposition

$$\sum_{n=0}^{\infty} c_{r,n} t^n = -\sum_{\zeta=\zeta_0} \frac{p^{r+1}\zeta t}{(1-pt)^{p^r} - \zeta} = -\sum_{\zeta=\zeta_r} \frac{\zeta}{1-\alpha t} = \sum_{n=0}^{\infty} \sum_{\zeta=\zeta_r} -\zeta \alpha^n t^n \tag{2.7}$$

where $\alpha = p/(1-\zeta)$ and the sums indexed by ζ_r are taken over *all* primitive p^{r+1}-th roots of unity $\zeta = \zeta_r$. Therefore

$$c_{r,n} = -\sum_{\zeta=\zeta_r} \zeta \left(\frac{p}{1-\zeta}\right)^n = -\mathrm{Tr}_r\left(\zeta_r \left(\frac{p}{1-\zeta_r}\right)^n\right) \tag{2.8}$$

for all r, n, where ζ_r denotes any fixed primitive p^{r+1}-th root of unity. This completes the proof of the b_n formula.

Dividing (2.4) by $1-pt$ yields

$$\frac{-pt}{(1-pt)\log(1-pt)} = \frac{1}{1-pt} - \sum_{r=0}^{\infty} \sum_{\zeta=\zeta_0} \frac{p^{r+1}\zeta t}{(1-pt)((1-pt)^{p^r} - \zeta)} \tag{2.9}$$

For any r and any primitive p-th root of unity $\zeta = \zeta_0$ we have the partial fraction decomposition

$$\frac{p^{r+1}\zeta_0 t}{(1-pt)((1-pt)^{p^r} - \zeta_0)} = \frac{-p^r}{1-pt} + \sum_{\alpha} \frac{1}{1-\alpha t} \tag{2.10}$$

where the sum indexed by α is over all p^r values of $\alpha = p/(1-\zeta_r)$ where ζ_r is a primitive p^{r+1}-th root of unity satisfying $\zeta_r^{p^r} = \zeta_0$. So in the partial fraction decomposition of the right side of (2.9) the terms with denominator $1-pt$ have numerators which sum to $1 + (p-1)\sum_{r=0}^{\infty} p^r$, which is zero in $\mathbb{Z}_p$, so that (2.9) becomes

$$\frac{-pt}{(1-pt)\log(1-pt)} = -\sum_{r=0}^{\infty} \sum_{\zeta=\zeta_r} \frac{1}{(1-\alpha t)} \tag{2.11}$$

where the sum indexed by ζ_r is now over *all* primitive p^{r+1}-th roots of unity $\zeta = \zeta_r$, with $\alpha = p/(1-\zeta)$. Expanding as power series we write (2.11) as

$$\sum_{n=0}^{\infty} \frac{(-p)^n B_n^{(n)} t^n}{n!} = \sum_{r=0}^{\infty} \sum_{n=0}^{\infty} C_{r,n} t^n \tag{2.12}$$

where

$$\sum_{n=0}^{\infty} C_{r,n} t^n = \sum_{\zeta=\zeta_r} \frac{-1}{(1-\alpha t)} = \sum_{n=0}^{\infty} \sum_{\zeta=\zeta_r} -\alpha^n t^n, \tag{2.13}$$

so that

$$C_{r,n} = -\sum_{\zeta=\zeta_r} \alpha^n = -\mathrm{Tr}_r\left(\left(\frac{p}{1-\zeta_r}\right)^n\right). \tag{2.14}$$

Since $(-p)^n B_n^{(n)}/n! = \sum_{r=0}^{\infty} C_{r,n}$ for all n, the result for $B_n^{(n)}$ follows.

Remarks. Since each trace map is an additive group homomorphism we may remove the factors of $(-p)^n$ from the congruences of these theorems and write

$$\frac{B_n^{(n)}}{n!} = -\sum_{r=0}^{\infty} \mathrm{Tr}_r\left((\zeta_r - 1)^{-n}\right) \tag{2.15}$$

and

$$b_n = -\sum_{r=0}^{\infty} \mathrm{Tr}_r\left(\zeta_r(\zeta_r - 1)^{-n}\right) \tag{2.16}$$

for all positive integers n; although the terms in these sums are not all integers, the sums still converge p-adically.

3. Application to Congruences.

Equations (2.6), (2.13) show that each sequence $\{C_{r,n}\}_{n=0}^{\infty}$ and $\{c_{r,n}\}_{n=0}^{\infty}$ satisfies a linear recurrence, with integer coefficients, of order $p^r(p-1)$. Since $(-p)^n B_n^{(n)}/n! = \sum_{r=0}^{\infty} C_{r,n}$ and $(-p)^n b_n = \sum_{r=0}^{\infty} c_{r,n}$ in $\mathbb{Z}_p$ we can get information about b_n and $B_n^{(n)}$ by analyzing these recurrent sequences.

Proposition. *With $c_{r,n}$ and $C_{r,n}$ as defined in (2.6), (2.13), respectively, we have*

$$\mathrm{ord} C_{r,n} \geq r + n - \left\lfloor \frac{n}{p^r(p-1)} \right\rfloor, \qquad \mathrm{ord} c_{r,n} \geq r + n - \left\lfloor \frac{n + p^r - 1}{p^r(p-1)} \right\rfloor$$

for all positive integers r and n.

Proof. For any primitive p-th root of unity ζ_0 the polynomial $P(t) = ((1-pt)^{p^r} - \zeta_0)/(1-\zeta_0)$ is a unit in the power series ring $\mathbb{Z}_p[\zeta_0][[t]]$, whose reciprocal roots $\alpha = p/(1-\zeta_r)$ all have p-adic ordinal $1 - 1/(p^r(p-1))$. If we introduce a change of variables $u = pt/(1-\zeta_r)$ then $P(t) \in \mathfrak{O}_r[u]$ with constant term 1, so that $P(t)$ is a unit in $\mathfrak{O}_r[[u]]$. It follows that $P(t)^{-1} = \sum a_{r,n} t^n \in \mathfrak{O}_r[[t]]$ with $\mathrm{ord} a_{r,n} \geq n(1 - 1/(p^r(p-1)))$. By the first equality of (2.6), $\sum c_{r,n} t^n$ is a sum of $p-1$ power series of the form $p^{r+1}(1-\zeta_0)^{-1} t \cdot P(t)^{-1}$, so that

$$\mathrm{ord} c_{r,n} \geq \left\lceil r + 1 - \frac{1}{p-1} + (n-1)\left(1 - \frac{1}{p^r(p-1)}\right) \right\rceil. \tag{3.1}$$

The proposition for $c_{r,n}$ follows by observing that $\lceil m+x\rceil = m+\lceil x\rceil$ for $m\in\mathbb{Z}$, $x\in\mathbb{R}$ and that $\lceil -x\rceil = -[x]$. To get the corresponding statement for $C_{r,n}$, begin by rewriting (2.10) as

$$\frac{p^{r+1}\zeta_0 t}{(1-pt)((1-pt)^{p^r}-\zeta_0)} = \frac{-p^r}{1-pt} + \frac{p^r((1-pt)^{p^r-1}-\zeta_0)}{(1-pt)^{p^r}-\zeta_0} \tag{3.2}$$

so that

$$\sum_{n=0}^{\infty} C_{r,n}t^n = \sum_{\zeta=\zeta_0} \frac{-p^r((1-pt)^{p^r-1}-\zeta_0)}{(1-pt)^{p^r}-\zeta_0}, \tag{3.3}$$

and proceed in the same way.

Corollary 1. *For all positive integers n we have*

$$\operatorname{ord}_p \frac{B_n^{(n)}}{n!} \geq -\left\lfloor \frac{n}{p-1}\right\rfloor$$

and

$$\frac{B_n^{(n)}}{n!} \equiv -\operatorname{Tr}_0\left((\zeta_0-1)^{-n}\right) \qquad (\operatorname{mod} p^{1-\lfloor n/(p^2-p)\rfloor}\mathbb{Z}_p);$$

furthermore,

$$\operatorname{ord}_p b_n \geq -\left\lfloor \frac{n}{p-1}\right\rfloor$$

and

$$b_n \equiv -\operatorname{Tr}_0\left(\zeta_0(\zeta_0-1)^{-n}\right) \qquad (\operatorname{mod} p^{1-\lfloor (n+p-1)/(p^2-p)\rfloor}\mathbb{Z}_p).$$

Proof. We have $(-p)^n B_n^{(n)}/n! \in C_{0,n}\mathbb{Z}_p$ and $(-p)^n b_n \in c_{0,n}\mathbb{Z}_p$, giving the first and third statements; similarly, $(-p)^n B_n^{(n)}/n! \equiv C_{0,n}$ (mod $C_{1,n}\mathbb{Z}_p$) and $(-p)^n b_n \equiv c_{0,n}$ (mod $c_{1,n}\mathbb{Z}_p$) give the remaining statements.

Remarks. The fact that $\operatorname{ord}_p(B_n^{(n)}/n!) \geq -\lfloor n/(p-1)\rfloor$ and $\operatorname{ord}_p b_n \geq -\lfloor n/(p-1)\rfloor$ has already been proved by Howard ([4], Theorem 5.1 and eq. (6.5)). These inequalities are generically best possible in the sense that for every prime p we have equality for infinitely many n (cf. [4], Theorem 5.2 and eqs. (6.9)-(6.12)); however, Howard's results also show that the inequalities are also strict for infinitely many n.

In §5,6 of [4] Howard gave congruences modulo 8, 9, and p for $p^{\lfloor n/(p-1)\rfloor}B_n^{(n)}/n!$ and for $p^{\lfloor n/(p-1)\rfloor}b_n$. Corollary 1 extends those congruences to congruences modulo p^{A_n} where $A_n = \lfloor n/(p-1)\rfloor + 1 - \lfloor n/(p^2-p)\rfloor$ for the $B_n^{(n)}/n!$ case and $A_n = \lfloor n/(p-1)\rfloor + 1 - \lfloor (n+p-1)/(p^2-p)\rfloor$

for the b_n case. That is, this corollary determines $B_n^{(n)}/n!$ and b_n accurate to roughly n/p p-adic digits instead of just one or two.

This expression for $B_n^{(n)}/n!$ and for b_n allows us to state systems of congruences for these sequences which resemble the Kummer congruences for the usual Bernoulli numbers $B_n = B_n^{(1)}$. For a sequence $\{a_m\}$ and a nonnegative integer c, we define the action of the forward difference operator Δ_c with increment c by

$$\Delta_c\, a_m = a_{m+c} - a_m. \tag{3.4}$$

The powers Δ_c^k of Δ_c are defined by $\Delta_c^0 =$ identity and $\Delta_c^k = \Delta_c \circ \Delta_c^{k-1}$ for positive integers k, so that

$$\Delta_c^k\, a_m = \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} a_{m+jc} \tag{3.5}$$

for all nonnegative integers k.

Corollary 2. *If $c \equiv 0$ (mod $p^a(p-1)$) then for all positive integers m we have*

$$\Delta_c^k \left\{ (-p)^{\lfloor \frac{m}{p-1} \rfloor} \frac{B_m^{(m)}}{m!} \right\} \equiv 0 \qquad (\mathrm{mod}\ p^A \mathbb{Z}_p),$$

where $m = r(p-1) + t$ with $0 \le t < p-1$ and $A = \min\{r + 1 - \lfloor r/p \rfloor, ak + \lceil (k-t)/(p-1) \rceil\}$; furthermore,

$$\Delta_c^k \left\{ (-p)^{\lfloor \frac{m}{p-1} \rfloor} b_m \right\} \equiv 0 \qquad (\mathrm{mod}\ p^{A'} \mathbb{Z}_p),$$

where $A' = \min\{r + 1 - \lfloor (r+1)/p \rfloor, ak + \lceil (k-t)/(p-1) \rceil\}$.

Proof. Let $\pi \in \mathfrak{O}_0$ denote a fixed solution to $\pi^{p-1} = -p$. From Corollary 1 and the definition of the trace map Tr_0 we have

$$\pi^n \frac{B_n^{(n)}}{n!} \equiv -\sum_{x \in S} x^n \quad (\mathrm{mod}\ \pi^n p^{1 - \lfloor n/(p^2-p) \rfloor} \mathfrak{O}_0) \tag{3.6}$$

where $S = \{\pi/(\zeta - 1) : \zeta = \zeta_0\}$. If $x \in S$ then x is a unit in $\mathfrak{O}_0$, so $x^{p-1} \equiv 1$ (mod $\pi \mathfrak{O}_0$) and by induction on a we have $x^{(p-1)p^a} \equiv 1$ (mod $\pi p^a \mathfrak{O}_0$). Therefore we have a congruence

$$\begin{aligned} \Delta_c^k \left\{ \pi^m \frac{B_m^{(m)}}{m!} \right\} &\equiv \Delta_c^k \left\{ -\sum_{x \in S} x^m \right\} = -\sum_{x \in S} x^m (x^c - 1)^k \\ &\equiv 0 \quad (\mathrm{mod}\ (\pi^m p^{1 - \lfloor m/(p^2-p) \rfloor}, \pi^k p^{ak})) \end{aligned} \tag{3.7}$$

in the ring $\mathfrak{O}_0$. Dividing by π^t yields

$$\Delta_c^k \left\{ (-p)^{\lfloor \frac{m}{p-1} \rfloor} \frac{B_m^{(m)}}{m!} \right\} \equiv 0 \qquad (\text{mod } (p^{r+1-\lfloor r/p \rfloor}, \pi^{k-t} p^{ak})), \tag{3.8}$$

but the left side is now clearly a rational number and thus the congruence is in $\mathbb{Z}_p$. The proof of the congruences for the b_n is entirely similar.

Remarks. The case $k = 1$ of these congruences for $B_n^{(n)}$ may be written as

$$(-p)^r \frac{B_m^{(m)}}{m!} \equiv (-p)^s \frac{B_n^{(n)}}{n!} \quad (\text{mod } p^A \mathbb{Z}_p) \tag{3.9}$$

where $m = r(p-1) + t$ and $n = s(p-1) + t$, and we have $A \geq 1$ in all cases except when $a = 0$ and $t \neq 0$. For $0 \leq t \leq 4$ these congruences modulo p may be deduced from ([4], Theorem 5.2). Similarly, one may deduce the b_n congruences modulo p from ([4], eqs. (6.9)-(6.12)) in the case $k = 1$, $a > 0$, and $1 \leq t \leq 4$.

Our method can also be used to reveal some special values of n for which these inequalities can be strengthened:

Theorem 2. *If $n = kp^{r+1}$ with k odd then*

$$\mathrm{Tr}_r \left(\left(\frac{p}{1 - \zeta_r} \right)^n \right) = 0;$$

if $n = kp^{r+1} + 2$ with k odd then

$$\mathrm{Tr}_r \left(\zeta_r \left(\frac{p}{1 - \zeta_r} \right)^n \right) = 0.$$

Proof. If $\zeta = \zeta_r$ denotes a primitive p^{r+1}-th root of unity and $\alpha = p/(1 - \zeta)$, then $-\alpha\zeta = \overline{\alpha}$ and therefore $\alpha^2(-\zeta) = |\alpha|^2$ (where $\overline{\alpha}$ denotes complex conjugate and $|\alpha|$ denotes complex absolute value). It follows that $\omega = \alpha/|\alpha|$ is a primitive $4p^{r+1}$-th root of unity, since in fact $\omega^{-2} = -\zeta$. We may therefore rewrite (2.14), (2.8) as

$$C_{r,n} = -\mathrm{Tr}_r(\alpha^n) = -\sum_{\zeta} \alpha^n = \sum_{\alpha} |\alpha|^n \omega^n \tag{3.10}$$

and

$$c_{r,n} = -\mathrm{Tr}_r(\zeta\alpha^n) = -\sum_{\zeta} \zeta\alpha^n = \sum_{\alpha} |\alpha|^n \omega^{n-2} \tag{3.11}$$

where the latter sums are over all values of $\alpha = p/(1-\zeta)$ for primitive p^{r+1}-th roots of unity ζ, with $\omega = \alpha/|\alpha|$. By pairing each such α with its complex conjugate we may write

$$C_{r,n} = \sum_{\alpha} |\alpha|^n (\omega^n + \overline{\omega}^n) \tag{3.12}$$

and

$$c_{r,n} = \sum_{\alpha} |\alpha|^n (\omega^{n-2} + \overline{\omega}^{n-2}) \tag{3.13}$$

where the sums are now over all such values of α with positive imaginary part. Since each ω is a primitive $4p^{r+1}$-th root of unity, it follows that $\omega^m = \pm i$, and therefore $\omega^m + \overline{\omega}^m = 0$, whenever m is an odd multiple of p^{r+1}. This completes the proof of the theorem.

Corollary 3. *For $n = kp^r$ with k odd we have*

$$\mathrm{ord}_p \frac{B_n^{(n)}}{n!} \geq r - \left\lfloor \frac{k}{p-1} \right\rfloor$$

and

$$\frac{B_n^{(n)}}{n!} \equiv -\mathrm{Tr}_r\left((\zeta_r - 1)^{-n}\right) \qquad (\mathrm{mod}\ p^{r+1-\lfloor k/(p^2-p)\rfloor}\mathbb{Z}_p);$$

furthermore, for $n = kp^r + 2$ with k odd we have

$$\mathrm{ord}_p b_n \geq r - \left\lfloor \frac{k+1}{p-1} \right\rfloor$$

and

$$b_n \equiv -\mathrm{Tr}_r\left(\zeta_r(\zeta_r - 1)^{-n}\right) \qquad (\mathrm{mod}\ p^{r+1-\lfloor (k+p)/(p^2-p)\rfloor}\mathbb{Z}_p).$$

Proof. Under the stated hypotheses we have $(-p)^n B_n^{(n)}/n! \in C_{r,n}\mathbb{Z}_p$ and $(-p)^n b_n \in c_{r,n}\mathbb{Z}_p$, giving the first and third statements; similarly, $(-p)^n B_n^{(n)}/n! \equiv C_{r,n}\ (\mathrm{mod}\ C_{r+1,n}\mathbb{Z}_p)$ and $(-p)^n b_n \equiv c_{r,n}$ $(\mathrm{mod}\ c_{r+1,n}\mathbb{Z}_p)$ give the remaining statements.

We conclude by stating an explicit version of Corollary 1 in the case $p = 3$. In this case the sequences $\{c_{1,n}\}$ and $\{C_{1,n}\}$ satisfy the second-order linear recurrence $a_n = 3a_{n-1} - 3a_{n-2}$, which is easy to analyze. We remark that for $p = 3$ the lower bound $\mathrm{ord}_p(B_n^{(n)}/n!) \geq -\lfloor n/(p-1)\rfloor$ is exact if and only if $n \not\equiv 3\ (\mathrm{mod}\ 6)$, and the lower bound $\mathrm{ord}_p b_n \geq -\lfloor n/(p-1)\rfloor$ is exact if and only if $n \not\equiv 5\ (\mathrm{mod}\ 6)$.

Corollary 4. *For all $n \geq 0$ we have*

$$\frac{B_n^{(n)}}{n!} \equiv 3^{-\lfloor n/2 \rfloor}(-1)^{m+1}\varepsilon_n \qquad (\mathrm{mod}\ 3^{1-m}\mathbb{Z}_3)$$

where $m = \lfloor n/6 \rfloor$ and ε_n has the value $2, -1, 1, 0, -1, 1$ according to whether $n \equiv 0, 1, 2, 3, 4, 5$ (mod 6).

For all $n \geq 0$ we have

$$b_n \equiv 3^{-\lfloor n/2 \rfloor}(-1)^m \eta_n \qquad (\mathrm{mod}\ 3^{1-m}\mathbb{Z}_3)$$

where $m = \lfloor (n+2)/6 \rfloor$ and η_n has the value $1, -1, 2, -1, -1, 0$ according to whether $n \equiv 0, 1, 2, 3, 4, 5$ (mod 6).

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