A \( p \)-adic formula for the Nörlund numbers

and for Bernoulli numbers of the second kind

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Abstract

We give formulas expressing the Nörlund numbers and the Bernoulli numbers of the second kind as \( p \)-adically convergent sums of traces of algebraic integers for odd primes \( p \). We use these formulas to derive new congruences and divisibility results for these sequences, including analogues of Kummer’s congruences.

Keywords: Nörlund numbers, Bernoulli numbers of second kind, Cauchy numbers, congruences, \( p \)-adic analysis

1. Introduction

The Bernoulli numbers of order \( w \), \( B_n^{(w)} \), are the rational numbers defined [8] by the generating function

\[
\left( \frac{t}{e^t - 1} \right)^w = \sum_{n=0}^{\infty} B_n^{(w)} \frac{t^n}{n!}.
\]

(1.1)

For \( n = w \) the numbers \( B_n^{(n)} \) are called Nörlund numbers [4], or Cauchy numbers of the second type ([3], [7]), and may also be determined by the generating function

\[
\frac{t}{(1 + t) \log(1 + t)} = \sum_{n=0}^{\infty} B_n^{(n)} \frac{t^n}{n!}
\]

(1.2)

(cf. [8]). The first few values are \( B_0^{(0)} = 1, B_1^{(1)} = -1/2, B_2^{(2)} = 5/6, B_3^{(3)} = -9/4, B_4^{(4)} = 251/30, B_5^{(5)} = -475/12 \). One important role they play in combinatorial analysis is through the formula

\[
B_n^{(n)} = \int_0^1 (x - 1)(x - 2) \cdots (x - n) \, dx
\]

(1.3)

(cf. [8]). The Bernoulli numbers of the second kind \( b_n \) are the rational numbers determined ([2], [5]) by the generating function

\[
\frac{t}{\log(1 + t)} = \sum_{n=0}^{\infty} b_n t^n.
\]

(1.4)

The numbers \( n! b_n \) have also been called Cauchy numbers of the first type ([3], [7]), and may be defined by

\[
n! b_n = \int_0^1 x(x - 1)(x - 2) \cdots (x - n + 1) \, dx.
\]

(1.5)
The first few values are $b_0 = 1, b_1 = 1/2, b_2 = -1/12, b_3 = 1/24, b_4 = -19/720, b_5 = 3/160$. These
sequences are related by the formulas [4]

$$
\frac{B_n^{(n)}}{n!} = \sum_{j=0}^{n} (-1)^{n-j} b_j \quad \text{and} \quad b_n = \frac{B_n^{(n)}}{n!} + \frac{B_{n-1}^{(n-1)}}{(n-1)!}.
$$  \ \ (1.6)

The main results of this paper are the following expressions of these sequences as $p$-adically
convergent sums of traces of certain algebraic integers:

**Theorem 1.** Let $p$ be an odd prime and for each $r \geq 0$ let $\zeta_r$ denote any primitive $p^{r+1}$-th root of
unity. Then for all nonnegative integers $n$ we have

$$
(-p)^n \frac{B_n^{(n)}}{n!} = -\sum_{r=0}^{\infty} \text{Tr}_r \left( \left( \frac{p}{1-\zeta_r} \right)^n \right)
$$

as a $p$-adically convergent sum of integers, where $\text{Tr}_r$ denotes the trace map from $\mathbb{Q}((\zeta_r))$ to $\mathbb{Q}$, and
for all positive integers $n$ we have

$$
(-p)^n b_n = -\sum_{r=0}^{\infty} \text{Tr}_r \left( \zeta_r \left( \frac{p}{1-\zeta_r} \right)^n \right)
$$

as a $p$-adically convergent sum of integers.

A similar formula for $p = 2$ was given in [9], where it was used to prove the conjectures of
Adelberg [1] concerning the 2-adic digits of $B_n^{(n)}/n!$ and of $b_n$. In Section 3 we apply Theorem 1
to prove new congruences for these sequences, including a version of Kummer congruences.

**2. Proof of $p$-adic formula.**

Throughout this paper $p$ will denote an odd prime, $\mathbb{Z}_p$ the ring of $p$-adic integers, $\mathbb{Q}_p$ the field
of $p$-adic numbers, and $\text{ord}_p$ the $p$-adic valuation normalized by $\text{ord}_p p = 1$. Clearly $\zeta = \zeta_r$ is
a primitive $p^{r+1}$-th root of unity if and only if $\zeta^{p^r} = \zeta_0$ is a primitive $p$-th root of unity, so the
minimal polynomial for $\zeta_r$ over $\mathbb{Q}$ is the $p^{r+1}$-th cyclotomic polynomial $\Phi_{p^{r+1}}(t) = \Phi_p(t^{p^r})$, where
$\Phi_p(t) = 1 + t + \cdots + t^{p-1}$ is the $p$-th cyclotomic polynomial. It is well known that $\mathbb{Q}(\zeta_r)$ is a
cyclic extension of $\mathbb{Q}$ with Galois group $\text{Gal}(\mathbb{Q}(\zeta_r)/\mathbb{Q}) = \{ \sigma_j : j \in (\mathbb{Z}/p^{r+1}\mathbb{Z})^\times \} \cong (\mathbb{Z}/p^{r+1}\mathbb{Z})^\times$, where $\sigma_j$ is the automorphism of $\mathbb{Q}(\zeta_r)$ induced by $\zeta_r \mapsto \zeta_r^j$. Since $1 - \zeta_0$ is a root of the $p$-
Eisenstein polynomial $\Phi_p(1 - t) = p - (\frac{p}{2}) t + \cdots + t^{p-1}$ we have $\text{ord}_p(1 - \zeta_0) = 1/(p - 1)$, and
thus $\text{ord}_p(1 - \zeta_r) = 1/(p^r(p - 1))$ for all $r$; hence the extension $K_r = \mathbb{Q}_p(\zeta_r)$ of $\mathbb{Q}_p$ is totally
ramified of degree $p^r(p-1)$, with ring of integers $\mathcal{O}_r = \{x \in K_r : \text{ord} x \geq 0\}$, maximal ideal $\mathfrak{p}_r = \{x \in K_r : \text{ord} x \geq 1/(p^r(p-1))\}$, and residue class field $\mathcal{O}_r/\mathfrak{p}_r$ isomorphic to $\mathbb{Z}/p\mathbb{Z}$ (cf. [6]).

**Proof of Theorem 1.** We begin with the partial fraction decomposition

$$
\frac{p}{T^p - 1} - \frac{1}{T - 1} = \sum_{\zeta = \zeta_0} \frac{\zeta}{T - \zeta},
$$

where the sum is over all primitive $p$-th roots of unity $\zeta = \zeta_0$. We substitute $T = (1 - pt)^{p^r}$ and multiply by $-p^{r+1}t$ to obtain

$$
\frac{-p^{r+2}t}{(1 - pt)^{p^{r+1}} - 1} - \frac{-p^{r+1}t}{(1 - pt)^{p^r} - 1} = \sum_{\zeta = \zeta_0} \frac{p^{r+1}t}{(1 - pt)^{p^r} - \zeta}.
$$

Note that the sum on the right in (2.2) is a power series with integer coefficients, since this is certainly true on the left. If we sum this equation from $r = 0$ to $r = s$, the left side telescopes, yielding

$$
\frac{-p^{s+2}t}{(1 - pt)^{p^{s+1}} - 1} - 1 = \sum_{r=0}^{s} \sum_{\zeta = \zeta_0} \frac{p^{r+1}t}{(1 - pt)^{p^r} - \zeta}.
$$

Since $((1 - pt)^{p^r} - \zeta)/(1 - \zeta)$ is a unit in the power series ring $\mathbb{Z}_p[[t]]$, the $r$-th term in the sum indexed by $r$ in (2.3) lies in $p^{r+1}\mathbb{Z}_p[[t]]$, hence the $p$-adic limit of partial sums as $s \to \infty$ exists.

Since $((1 + t)^a - 1)/a \to \log(1 + t)$ as $a \to 0$, we have

$$
\frac{-pt}{\log(1 - pt)} - 1 = \sum_{r=0}^{\infty} \sum_{\zeta = \zeta_0} \frac{p^{r+1}t}{(1 - pt)^{p^r} - \zeta}.
$$

(2.4)

as an identity in $\mathbb{Z}[[t]]$. Expanding the left side of (2.4) as a power series gives

$$
\sum_{n=0}^{\infty} (-p)^n b_n t^n = 1 - \sum_{r=0}^{\infty} \sum_{\zeta = \zeta_0} \frac{p^{r+1}t}{(1 - pt)^{p^r} - \zeta}.
$$

(2.5)

We define rational integers $c_{r,n}$ by

$$
\sum_{n=0}^{\infty} c_{r,n} t^n = - \sum_{\zeta = \zeta_0} \frac{p^{r+1}t}{(1 - pt)^{p^r} - \zeta},
$$

(2.6)

so that $(-p)^n b_n = \sum_{r=0}^{\infty} c_{r,n}$ as a convergent sum in $\mathbb{Z}_p$ for all $n > 0$.

If $Q(t) = \prod_{i=1}^{n} (1 - \alpha_i t)$ is a polynomial of degree $n$ with distinct roots and $P(t)$ is a polynomial of degree less than $n$, it is easily seen that there is a partial fraction decomposition of $P(t)/Q(t)$
as $\sum_{i=1}^{n} a_i/(1 - \alpha_i t)$ where $a_i = -\alpha_i P(\alpha_i^{-1})/Q'(\alpha_i^{-1})$ for all $i$. Thus since $(1 - pt)^{\nu} - \zeta_0 = 0$ whenever $pt = 1 - \zeta_r$ for any of the $p^r$ primitive $p^{r+1}$-th roots of unity $\zeta_r$ satisfying $\zeta_r^{p^r} = \zeta_0$, we have by partial fraction decomposition

$$\sum_{n=0}^{\infty} c_{r,n} t^n = -\sum_{\zeta = \zeta_0} \frac{p^{r+1} \zeta t}{(1 - pt)^{\nu} - \zeta} = -\sum_{\zeta = \zeta_r} \frac{\zeta}{1 - \alpha t} = \sum_{n=0}^{\infty} \sum_{\zeta = \zeta_r} -\zeta \alpha^n t^n$$  (2.7)

where $\alpha = p/(1 - \zeta)$ and the sums indexed by $\zeta_r$ are taken over all primitive $p^{r+1}$-th roots of unity $\zeta = \zeta_r$. Therefore

$$c_{r,n} = -\sum_{\zeta = \zeta_r} \zeta \left( \frac{p}{1 - \zeta} \right)^n = -\text{Tr}_r \left( \zeta_r \left( \frac{p}{1 - \zeta_r} \right)^n \right)$$  (2.8)

for all $r, n$, where $\zeta_r$ denotes any fixed primitive $p^{r+1}$-th root of unity. This completes the proof of the $b_n$ formula.

Dividing (2.4) by $1 - pt$ yields

$$-\frac{pt}{(1 - pt) \log(1 - pt)} = \frac{1}{1 - pt} - \sum_{r=0}^{\infty} \sum_{\zeta = \zeta_0} \frac{p^{r+1} \zeta t}{(1 - pt)^{\nu} - \zeta}$$  (2.9)

For any $r$ and any primitive $p$-th root of unity $\zeta = \zeta_0$ we have the partial fraction decomposition

$$\frac{p^{r+1} \zeta_0 t}{(1 - pt)^{\nu} - \zeta_0} = -\frac{p^r}{1 - pt} + \sum_{\alpha} \frac{1}{1 - \alpha t}$$  (2.10)

where the sum indexed by $\alpha$ is over all $p^r$ values of $\alpha = p/(1 - \zeta_r)$ where $\zeta_r$ is a primitive $p^{r+1}$-th root of unity satisfying $\zeta_r^{p^r} = \zeta_0$. So in the partial fraction decomposition of the right side of (2.9) the terms with denominator $1 - pt$ have numerators which sum to $1 + (p - 1) \sum_{r=0}^{\infty} p^r$, which is zero in $\mathbb{Z}_p$, so that (2.9) becomes

$$-\frac{pt}{(1 - pt) \log(1 - pt)} = -\sum_{r=0}^{\infty} \sum_{\zeta = \zeta_r} \frac{1}{(1 - \alpha t)}$$  (2.11)

where the sum indexed by $\zeta_r$ is now over all primitive $p^{r+1}$-th roots of unity $\zeta = \zeta_r$, with $\alpha = p/(1 - \zeta)$. Expanding as power series we write (2.11) as

$$\sum_{n=0}^{\infty} \frac{(-p)^n B^{(n)}_{\nu} t^n}{n!} = \sum_{r=0}^{\infty} \sum_{n=0}^{\infty} c_{r,n} t^n$$  (2.12)

where

$$\sum_{n=0}^{\infty} c_{r,n} t^n = \sum_{\zeta = \zeta_r} \frac{-1}{1 - \alpha t} = \sum_{n=0}^{\infty} \sum_{\zeta = \zeta_r} -\alpha^n t^n,$$  (2.13)

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so that
\[ C_{r,n} = - \sum_{\zeta = \zeta_r} \alpha^n = - \text{Tr}_r \left( \left( \frac{p}{1 - \zeta_r} \right)^n \right). \] (2.14)

Since \((-p)^n B_r^{(n)} / n! = \sum_{r=0}^{\infty} C_{r,n}\) for all \(n\), the result for \(B_r^{(n)}\) follows.

**Remarks.** Since each trace map is an additive group homomorphism we may remove the factors of \((-p)^n\) from the congruences of these theorems and write
\[ \frac{B_r^{(n)}}{n!} = - \sum_{r=0}^{\infty} \text{Tr}_r \left( (\zeta_r - 1)^{-n} \right) \] (2.15)

and
\[ b_n = - \sum_{r=0}^{\infty} \text{Tr}_r \left( \zeta_r(\zeta_r - 1)^{-n} \right) \] (2.16)

for all positive integers \(n\); although the terms in these sums are not all integers, the sums still converge \(p\)-adically.

3. Application to Congruences.

Equations (2.6), (2.13) show that each sequence \(\{C_{r,n}\}_{n=0}^{\infty}\) and \(\{c_{r,n}\}_{n=0}^{\infty}\) satisfies a linear recurrence, with integer coefficients, of order \(p^r(p-1)\). Since \((-p)^n B_r^{(n)} / n! = \sum_{r=0}^{\infty} C_{r,n}\) and \((-p)^n b_n = \sum_{r=0}^{\infty} c_{r,n}\) in \(\mathbb{Z}_p\) we can get information about \(b_n\) and \(B_r^{(n)}\) by analyzing these recurrent sequences.

**Proposition.** With \(c_{r,n}\) and \(C_{r,n}\) as defined in (2.6), (2.13), respectively, we have
\[ \text{ord} C_{r,n} \geq r + n - \left\lfloor \frac{n}{p^r(p-1)} \right\rfloor, \quad \text{ord} c_{r,n} \geq r + n - \left\lfloor \frac{n + p^r - 1}{p^r(p-1)} \right\rfloor \]
for all positive integers \(r\) and \(n\).

**Proof.** For any primitive \(p\)-th root of unity \(\zeta_0\) the polynomial \(P(t) = ((1 - pt)^{p^r} - \zeta_0)/(1 - \zeta_0)\) is a unit in the power series ring \(\mathbb{Z}_p[\zeta_0][[t]]\), whose reciprocal roots \(\alpha = p/(1 - \zeta_r)\) all have \(p\)-adic ordinal \(1 - 1/(p^r(p-1))\). If we introduce a change of variables \(u = pt/(1 - \zeta_r)\) then \(P(t) \in \mathcal{O}_r[[u]]\) with constant term 1, so that \(P(t)\) is a unit in \(\mathcal{O}_r[[u]]\). It follows that \(P(t)^{-1} = \sum a_{r,n} t^n \in \mathcal{O}_r[[t]]\) with \(\text{ord}_r a_{r,n} \geq n(1 - 1/(p^r(p-1)))\). By the first equality of (2.6), \(\sum c_{r,n} t^n\) is a sum of \(p - 1\) power series of the form \(p^{r+1}(1 - \zeta_0)^{-1} t \cdot P(t)^{-1}\), so that
\[ \text{ord}_r c_{r,n} \geq \left[ r + 1 - \frac{1}{p - 1} + (n - 1) \left( 1 - \frac{1}{p^r(p-1)} \right) \right]. \] (3.1)
The proposition for \( c_{r,n} \) follows by observing that \([m + x] = m + [x]\) for \( m \in \mathbb{Z}, x \in \mathbb{R} \) and that \([−x] = −[x]\). To get the corresponding statement for \( C_{r,n} \), begin by rewriting (2.10) as

\[
\frac{p^{r+1} t_0}{(1 - pt)((1 - pt)^p - ζ_0)} = - \frac{p^r}{1 - pt} + \frac{p^r((1 - pt)^p - 1)}{(1 - pt)^p - ζ_0} \tag{3.2}
\]

so that

\[
\sum_{n=0}^{∞} C_{r,n} t^n = \sum_{ζ = ζ_0} - \frac{p^r((1 - pt)^p - 1)}{(1 - pt)^p - ζ_0}, \tag{3.3}
\]

and proceed in the same way.

**Corollary 1.** For all positive integers \( n \) we have

\[
\text{ord}_p \frac{B_{n}^{(n)}}{n!} \geq - \left\lfloor \frac{n}{p-1} \right\rfloor
\]

and

\[
\frac{B_{n}^{(n)}}{n!} \equiv - \text{Tr}_0 \left((ζ_0 - 1)^{-n}\right) \pmod{p^{1 - \left\lfloor n/(p^2-p) \right\rfloor} \mathbb{Z}_p};
\]

furthermore,

\[
\text{ord}_p b_n \geq - \left\lfloor \frac{n}{p-1} \right\rfloor
\]

and

\[
b_n \equiv - \text{Tr}_0 \left(ζ_0(ζ_0 - 1)^{-n}\right) \pmod{p^{1 - \left\lfloor (n+p-1)/(p^2-p) \right\rfloor} \mathbb{Z}_p}.
\]

**Proof.** We have \((-p)^n B_{n}^{(n)} / n! \in C_{0,n} \mathbb{Z}_p \) and \((-p)^n b_n \in c_{0,n} \mathbb{Z}_p \), giving the first and third statements; similarly, \((-p)^n B_{n}^{(n)} / n! \equiv C_{0,n} (\mod C_{1,n} \mathbb{Z}_p) \) and \((-p)^n b_n \equiv c_{0,n} (\mod c_{1,n} \mathbb{Z}_p) \) give the remaining statements.

**Remarks.** The fact that \( \text{ord}_p (B_{n}^{(n)} / n!) \geq - \left\lfloor n/(p-1) \right\rfloor \) and \( \text{ord}_p b_n \geq - \left\lfloor n/(p-1) \right\rfloor \) has already been proved by Howard ([4], Theorem 5.1 and eq. (6.5)). These inequalities are generically best possible in the sense that for every prime \( p \) we have equality for infinitely many \( n \) (cf. [4], Theorem 5.2 and eqs. (6.9)-(6.12)); however, Howard’s results also show that the inequalities are also strict for infinitely many \( n \).

In §5.6 of [4] Howard gave congruences modulo 8, 9, and \( p \) for \( p^{n/(p-1)} B_{n}^{(n)} / n! \) and for \( p^{n/(p-1)} b_n \). Corollary 1 extends those congruences to congruences modulo \( p^{A_n} \) where \( A_n = \left[n/(p-1)\right] + 1 - \left[n/(p^2-p)\right] \) for the \( B_{n}^{(n)} / n! \) case and \( A_n = \left[n/(p-1)\right] + 1 - \left[(n+p-1)/(p^2-p)\right] \)
for the $b_n$ case. That is, this corollary determines $B^{(n)}_n / n!$ and $b_n$ accurate to roughly $n/p$ $p$-adic digits instead of just one or two.

This expression for $B^{(n)}_n / n!$ and for $b_n$ allows us to state systems of congruences for these sequences which resemble the Kummer congruences for the usual Bernoulli numbers $B_n = B^{(1)}_n$. For a sequence $\{a_m\}$ and a nonnegative integer $c$, we define the action of the forward difference operator $\Delta_c$ with increment $c$ by

$$\Delta_c a_m = a_{m+c} - a_m.$$  \hspace{1cm} (3.4)

The powers $\Delta^k_c$ of $\Delta_c$ are defined by $\Delta^0_c = \text{identity}$ and $\Delta^k_c = \Delta_c \circ \Delta^{k-1}_c$ for positive integers $k$, so that

$$\Delta^k_c a_m = \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} a_{m+jc}$$  \hspace{1cm} (3.5)

for all nonnegative integers $k$.

**Corollary 2.** If $c \equiv 0 \pmod{p^a(p-1)}$ then for all positive integers $m$ we have

$$\Delta^k_c \left \{ (-p)^{\frac{m}{p-1}} \frac{B^{(m)}_m}{m!} \right \} \equiv 0 \pmod{p^A \mathbb{Z}_p},$$

where $m = r(p-1) + t$ with $0 \leq t < p - 1$ and $A = \min \{ r + 1 - \lfloor r/p \rfloor, ak + \lceil (k - t)/(p - 1) \rceil \}$; furthermore,

$$\Delta^k_c \left \{ (-p)^{\frac{m}{p-1}} b_m \right \} \equiv 0 \pmod{p^{A'} \mathbb{Z}_p},$$

where $A' = \min \{ r + 1 - \lceil (r + 1)/p \rceil, ak + \lceil (k - t)/(p - 1) \rceil \}$.

**Proof.** Let $\pi \in \mathcal{O}_0$ denote a fixed solution to $\pi^{p-1} = -p$. From Corollary 1 and the definition of the trace map $\text{Tr}_0$ we have

$$\pi^n \frac{B^{(n)}_n}{n!} \equiv - \sum_{x \in S} x^n \pmod{\pi^{n^p - \lfloor n/(p^2-p) \rfloor} \mathcal{O}_0}$$  \hspace{1cm} (3.6)

where $S = \{ \pi/(\zeta - 1) : \zeta = \zeta_0 \}$. If $x \in S$ then $x$ is a unit in $\mathcal{O}_0$, so $x^{p-1} \equiv 1 \pmod{\pi \mathcal{O}_0}$ and by induction on $a$ we have $x^{(p-1)p^a} \equiv 1 \pmod{\pi^a \mathcal{O}_0}$. Therefore we have a congruence

$$\Delta^k_c \left \{ \pi^n \frac{B^{(m)}_m}{m!} \right \} \equiv \sum_{x \in S} x^m \equiv \sum_{x \in S} x^m(x - 1)^k$$  \hspace{1cm} (3.7)

$$\equiv 0 \pmod{\pi^{n^p - \lfloor n/(p^2-p) \rfloor}, \pi^k p^a},$$

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in the ring $\mathcal{O}_0$. Dividing by $\pi^t$ yields
\[
\Delta^k \left\{ (-p)^{\left[ \frac{m}{r} \right]} \frac{B_{m}^{(t)}}{m!} \right\} \equiv 0 \pmod{(p^{r+1} - |r/p|, \pi^{k-t} p^k)} ,
\]
but the left side is now clearly a rational number and thus the congruence is in $\mathbb{Z}_p$. The proof of
the congruences for the $b_n$ is entirely similar.

**Remarks.** The case $k = 1$ of these congruences for $B_{n}^{(n)}$ may be written as
\[
(-p)^r B_{m}^{(m)} \equiv (-p)^x B_{n}^{(n)} \pmod{p^A \mathbb{Z}_p}
\]
where $m = r(p - 1) + t$ and $n = s(p - 1) + t$, and we have $A \geq 1$ in all cases except when $a = 0$
and $t \neq 0$. For $0 \leq t \leq 4$ these congruences modulo $p$ may be deduced from ([4], Theorem 5.2).
Similarly, one may deduce the $b_n$ congruences modulo $p$ from ([4], eqs. (6.9)-(6.12)) in the case
$k = 1, a > 0$, and $1 \leq t \leq 4$.

Our method can also be used to reveal some special values of $n$ for which these inequalities
can be strengthened:

**Theorem 2.** If $n = kp^{r+1}$ with $k$ odd then
\[
\text{Tr}_r \left( \left( \frac{p}{1 - \zeta} \right)^n \right) = 0;
\]
if $n = kp^{r+1} + 2$ with $k$ odd then
\[
\text{Tr}_r \left( \zeta \left( \frac{p}{1 - \zeta} \right)^n \right) = 0.
\]

**Proof.** If $\zeta = \zeta_r$ denotes a primitive $p^{r+1}$-th root of unity and $\alpha = p/(1 - \zeta)$, then $-\alpha \zeta = \overline{\alpha}$ and
therefore $\alpha^2 (-\zeta) = |\alpha|^2$ (where $\overline{\alpha}$ denotes complex conjugate and $|\alpha|$ denotes complex absolute
value). It follows that $\omega = \alpha/|\alpha|$ is a primitive $4p^{r+1}$-th root of unity, since in fact $\omega^{-2} = -\zeta$. We
may therefore rewrite (2.14), (2.8) as
\[
C_{r,n} = -\text{Tr}_r (\alpha^n) = - \sum_{\zeta} \alpha^n = \sum_{\alpha} |\alpha|^n \omega^n
\]
and
\[
c_{r,n} = -\text{Tr}_r (\zeta \alpha^n) = - \sum_{\zeta} \zeta \alpha^n = \sum_{\alpha} |\alpha|^n \omega^{n-2}
\]

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where the latter sums are over all values of \( \alpha = p/(1 - \zeta) \) for primitive \( p^{r+1} \)-th roots of unity \( \zeta \), with \( \omega = \alpha/|\alpha| \). By pairing each such \( \alpha \) with its complex conjugate we may write

\[
C_r_n = \sum_{\alpha} |\alpha|^n (\omega^n + \overline{\omega}^n)
\]  

(3.12)

and

\[
c_r_n = \sum_{\alpha} |\alpha|^n (\omega^{n-2} + \overline{\omega}^{n-2})
\]  

(3.13)

where the sums are now over all such values of \( \alpha \) with positive imaginary part. Since each \( \omega \) is a primitive \( 4p^{r+1} \)-th root of unity, it follows that \( \omega^m = \pm i \), and therefore \( \omega^m + \overline{\omega}^m = 0 \), whenever \( m \) is an odd multiple of \( p^{r+1} \). This completes the proof of the theorem.

**Corollary 3.** For \( n = kp^r \) with \( k \) odd we have

\[
\text{ord}_p B_n^{(n)} \geq r - \left| \frac{k}{p - 1} \right|
\]

and

\[
\frac{B_n^{(n)}}{n!} \equiv -\text{Tr}_r \left( (\zeta_r - 1)^{-n} \right) \pmod{p^{r+1 - \lfloor k/(p^2 - p) \rfloor} \mathbb{Z}_p};
\]

furthermore, for \( n = kp^r + 2 \) with \( k \) odd we have

\[
\text{ord}_p b_n \geq r - \left| \frac{k + 1}{p - 1} \right|
\]

and

\[
b_n \equiv -\text{Tr}_r \left( \zeta_r (\zeta_r - 1)^{-n} \right) \pmod{p^{r+1 - \lfloor (k + p)/(p^2 - p) \rfloor} \mathbb{Z}_p}.
\]

**Proof.** Under the stated hypotheses we have \((-p)^n B_n^{(n)} / n! \in C_{r,n} \mathbb{Z}_p \) and \((-p)^n b_n \in c_{r,n} \mathbb{Z}_p \), giving the first and third statements; similarly, \((-p)^n B_n^{(n)} / n! \equiv c_{r,n} \pmod{c_{r+1,n} \mathbb{Z}_p} \) and \((-p)^n b_n \equiv c_{r,n} \pmod{c_{r+1,n} \mathbb{Z}_p} \) give the remaining statements.

We conclude by stating an explicit version of Corollary 1 in the case \( p = 3 \). In this case the sequences \( \{a_n\} \) and \( \{C_1,n\} \) satisfy the second-order linear recurrence \( a_n = 3a_{n-1} - 3a_{n-2} \), which is easy to analyze. We remark that for \( p = 3 \) the lower bound \( \text{ord}_p (B_n^{(n)} / n!) \geq -\lfloor n/(p - 1) \rfloor \) is exact if and only if \( n \not\equiv 3 \pmod{6} \), and the lower bound \( \text{ord}_p b_n \geq -\lfloor n/(p - 1) \rfloor \) is exact if and only if \( n \not\equiv 5 \pmod{6} \).
Corollary 4. For all \( n \geq 0 \) we have

\[
\frac{B_n^{(n)}}{n!} \equiv 3^{-\lfloor n/2 \rfloor}(-1)^{m+1} \varepsilon_n \pmod{3^{1-m} \mathbb{Z}_3}
\]

where \( m = \lfloor n/6 \rfloor \) and \( \varepsilon_n \) has the value 2, -1, 1, 0, -1, 1 according to whether \( n \equiv 0, 1, 2, 3, 4, 5 \pmod{6} \).

For all \( n \geq 0 \) we have

\[
b_n \equiv 3^{-\lfloor (n+2)/6 \rfloor}(-1)^m \eta_n \pmod{3^{1-m} \mathbb{Z}_3}
\]

where \( m = \lfloor (n+2)/6 \rfloor \) and \( \eta_n \) has the value 1, -1, 2, -1, -1, 0 according to whether \( n \equiv 0, 1, 2, 3, 4, 5 \pmod{6} \).

References


