On p-adic multiple zeta and log gamma functions

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Abstract

We define p-adic multiple zeta and log gamma functions using multiple Volkenborn integrals, and develop some of their properties. Although our functions are close analogues of classical Barnes multiple zeta and log gamma functions and have many properties similar to them, we find that our p-adic analogues also satisfy reflection functional equations which have no analogues to the complex case. We conclude with a Laurent series expansion of the p-adic multiple log gamma function for (p-adically) large x which agrees exactly with Barnes's asymptotic expansion for the (complex) multiple log gamma function, with the fortunate exception that the error term vanishes. Indeed, it was the possibility of such an expansion which served as the motivation for our functions, since we can use these expansions computationally to p-adically investigate conjectures of Gross, Kashio, and Yoshida over totally real number fields.

Key words: Barnes multiple zeta functions, multiple gamma functions, Diamond function, Volkenborn integral, p-adic analysis

1. Introduction

Over 100 years ago, E. W. Barnes [B] introduced a new class of zeta functions that extend and generalize the Hurwitz zeta function $\zeta(s,x)$ in a natural way. The multiple zeta function of Barnes, denoted by $\zeta_N(s,x;\omega_1,\ldots,\omega_N)$, depends upon several parameters $x,\omega_1,\ldots,\omega_N$, which may all be taken to be positive real numbers for now. These functions all have meromorphic continuations with respect to the variable s to the whole complex plane and are all analytic at $s=0$. When $N=1$ and $\omega_1=1$, we have $\zeta_1(s,x;1)=\zeta(s,x)$, and the following intimate relationship holds with the classical gamma function

$$\frac{\partial}{\partial s}\zeta(s,x)\bigg|_{s=0}=\log\left(\frac{\Gamma(x)}{\sqrt{2\pi}}\right). \tag{1.1}$$

Preprint submitted to Elsevier *January 5, 2011*

Barnes introduced a modular constant $\rho_N(\bar{\omega})$, where $\bar{\omega} = (\omega_1, \ldots, \omega_N)$, that plays a role analogous to $\sqrt{2\pi}$ in (1.1). He then defined ([B], §23) the multiple gamma function $\Gamma_N^B(x;\bar{\omega})$ by

$$\frac{\partial}{\partial s}\zeta_N(s,x;\bar{\omega})\bigg|_{s=0} = \log\left(\frac{\Gamma_N^B(x;\bar{\omega})}{\rho_N(\bar{\omega})}\right). \tag{1.2}$$

Following Ruijsenaars [R], we simply set $\Gamma_N(x;\bar{\omega}) = \Gamma_N^B(x;\bar{\omega})/\rho_N(\bar{\omega})$.

The multiple zeta and log gamma functions received very little attention during the first 70 years following their introduction until Shintani [Sh] gave a new and illuminating evaluation of the special values of certain L-functions attached to real quadratic number fields in terms of these functions of Barnes. Independently of Shintani and unaware of Barnes's work, Pierrette Cassou-Noguès [CN1] studied multiple zeta functions and developed p-adic versions of these functions as well, the latter realized in terms of multiple Volkenborn integrals. She also defined p-adic analogues of the multiple log gamma functions, first [CN2] in terms of the p-adic partial zeta functions associated to totally real number fields, and later [CN3] in terms of p-adic multiple zeta functions. More recently, Kashio [K] has revisited and refined some of the work of Shintani and Cassou-Noguès and obtained a remarkably close p-adic analogue to an important formula of Shintani.

In the meantime, more elementary approaches have been developed for the p-adic versions of the Hurwitz zeta function and logarithm of the classical gamma function by Washington [W] and Diamond [D], respectively. Diamond used the Volkenborn integral to define a p-adic function $G_p(x)$ which has many properties in common with the complex log gamma function; more precisely, it mirrors the exact expression appearing on the right side of Eq. (1.1). We will elaborate more fully on these properties later but wish to highlight one property in particular here. When $|x|_p > 1$, we have

$$G_p(x) = \left(x - \frac{1}{2}\right)\log_p x - x + \sum_{j=2}^{\infty}\frac{(-1)^j(j-2)!}{j!}B_j x^{1-j}, \tag{1.3}$$

where B_j is the jth Bernoulli number. The asymptotic formula (Stirling's theorem) for $\log(\Gamma(x)/\sqrt{2\pi})$ has exactly the same form as the right side of (1.3) except that the infinite sum in (1.3) must be replaced by a finite sum and an error term defined by an improper integral. The infinite sum in (1.3) does not converge in $\mathbb{C}$ but does converge p-adically! Formula (1.3) offers not only a straightforward and elegant connection to the complex case but also an efficient and easy-to-use formula for computing $G_p(x)$ as well. Barnes derived ([B], §57) an asymptotic formula for $\log(\Gamma_N(x;\bar{\omega}))$ which generalizes Stirling's theorem. Modern presentations of Barnes's asymptotic formula may be found in both [R] and [Yo]. In terms of the notation used in [R] (see our Section 2), we have

$$\begin{aligned}\log(\Gamma_N(x;\bar{\omega})) &= \frac{(-1)^{N+1}}{N!}B_{N,N}(x;\bar{\omega})\log x \\ &+ \sum_{j=0}^{N-1}\frac{(-1)^N}{j!(N-j)!}\left(1+\frac{1}{2}+\cdots+\frac{1}{N-j}\right)B_{N,j}(0;\bar{\omega})x^{N-j} \\ &+ \sum_{j=N+1}^{M}\frac{(-1)^j(j-N-1)!}{j!}B_{N,j}(0;\bar{\omega})x^{N-j}+R_{N,M}, \quad (1.4)\end{aligned}$$

where $R_{N,M}$ is an improper integral (see [R], Eq. (3.14), for its exact expression). The $B_{N,n}(x;\bar{\omega})$ appearing here are polynomials in several variables that restrict to the usual Bernoulli polynomials when $N = 1$ and $\omega_1 = 1$. The present paper began with the idea that (1.4) might produce a p-adically convergent series representation for the p-adic multiple log gamma function $G_{p,N}(x;\bar{\omega})$ in the same way that (1.3) relates Diamond's function to Stirling's asymptotic formula. The same advantages offered by Diamond's approach would then apply here rendering the function $G_{p,N}(x;\bar{\omega})$ more accessible and also giving an efficient formula for its computation as well. In Theorem 4.2, we prove that this idea works out exactly for appropriate parameter values. While developing this idea, we also discovered (see Theorem 4.1) a new representation for the p-adic multiple zeta function $\zeta_{p,N}(s,x;\bar{\omega})$ (again, for appropriate parameter values) that directly generalizes Washington's ([W], §5.2) Laurent series representation of the p-adic Hurwitz zeta function. We were also led to define the p-adic multiple zeta and log gamma functions themselves in a new and slightly modified form. We define these functions using multiple Volkenborn integrals just as Kashio does ([K], see also [KY]), but we use a slightly different integrand that we find easier to work with and which leads to p-adic functions sharing more properties and closer analogies with their complex-valued counterparts. Our definitions were inspired by and directly generalize Diamond's Volkenborn integral definition of $G_p(x)$. Over certain parameter ranges, our functions agree exactly with those defined by Kashio. Over other parameter ranges, our functions relate to the functions of [K] as the Diamond log gamma function G_p relates to the logarithm of the Morita gamma function (see Theorem 3.5).

In Section 2, we review several of the properties of the Barnes multiple zeta and log gamma functions that we use for comparison purposes later when we develop their p-adic counterparts. Some necessary p-adic analytic preliminaries are also discussed in Section 2.

In section 3, we establish several basic properties of the p-adic multiple zeta and log gamma functions, many of which are close analogues of the corresponding complex-valued functions. One notable exception is the reflection functional equation (Theorem 3.2(iii) and Theorem 3.4(iii)), which follows from a basic property of the Volkenborn integral but is not the p-adic analogue of a complex functional equation. The manner in which $\zeta_{p,N}(s,x;\bar{\omega})$ interpolates the values of $\zeta_N(s,x;\bar{\omega})$ when s is a nonpositive integer is given in Theorem 3.2(v). This crucial connection is the original *raison d'être* for the p-adic versions of these functions. The two main theorems of Section 4 were already discussed above. Section 5 covers a technical point relevant to the prime $p = 2$.

In [T], the first author used double gamma functions and a continued fraction algorithm to give expressions for Stark units over real quadratic fields. In a forthcoming paper [TY], we use Kashio's formula [K] to adapt those methods p-adically to computationally investigate a conjecture of Gross ([G], conjecture 3.13) over real quadratic fields. Our method in this case calls for the calculation of $G_{p,2}(x, \bar{\omega})$ values whose arguments always satisfy $x, \omega_1, \omega_2 \in \mathbb{Q}_p$ and $|x|_p > \max\{|\omega_1|_p, |\omega_2|_p\}$; Theorem 4.2 gives a very efficient formula for such calculations. Stark [St] has given a method for computing values of a p-adic log double gamma function but only under rather restrictive values allowed for the parameters.

2. Multiple zeta functions and Bernoulli polynomials of higher order

In the first part of this section, we review some of the basic properties of the multiple zeta and log gamma functions originally introduced by Barnes (the reader is referred to [R] for an excellent modern treatment). The higher order Bernoulli polynomials arise naturally in the study of these functions and in the latter half of this section we give a p-adic realization of these polynomials in terms of multiple Volkenborn integrals.

Let $\mathbb{Z}_0$ denote the set of nonnegative integers and $\mathbb{Z}^+$ the set of positive integers. Suppose $\omega_1, \ldots, \omega_N$ are positive real numbers and x is a complex number with positive real part. The *Barnes multiple zeta function* $\zeta_N(s, x; \bar{\omega})$ *with parameter vector* $\bar{\omega} = (\omega_1, \ldots, \omega_N)$ is defined for $\Re(s) > N$ by

$$\zeta_N(s, x; \omega_1, \ldots, w_N) = \sum_{t_1=0}^{\infty} \cdots \sum_{t_N=0}^{\infty} (x + \omega_1 t_1 + \cdots + \omega_N t_N)^{-s}, \tag{2.1}$$

which may also be written more concisely as

$$\zeta_N(s, x; \bar{\omega}) = \sum_{\bar{t} \in \mathbb{Z}_0^N} (x + \bar{\omega} \cdot \bar{t})^{-s}. \tag{2.2}$$

When $N = 1$ and $\omega_1 = 1$ this is the Hurwitz zeta function; yet one further specialization to $x = 1$ gives the Riemann zeta function. When $N = 0$ there is no parameter vector $\bar{\omega}$ but we may still regard the above equation as defining the function $\zeta_0(s, x; -) = x^{-s}$. As a function of s, $\zeta_N(s, x; \bar{\omega})$ is analytic for $\Re(s) > N$ and it has a meromorphic continuation to all of $\mathbb{C}$ with simple poles at $s = 1, \ldots, N$. The following properties also hold:

$$\zeta_N(s, x + \omega_N; \omega_1, \ldots, \omega_N) - \zeta_N(s, x; \omega_1, \ldots, \omega_N) = -\zeta_{N-1}(s, x; \omega_1, \ldots, \omega_{N-1}); \tag{2.3}$$

$$\zeta_N(s, cx; c\bar{\omega}) = c^{-s} \zeta_N(s, x; \bar{\omega}) \quad \text{for all } c \in \mathbb{R}^+; \tag{2.4}$$

$$\frac{\partial^m}{\partial x^m}\zeta_N(s,x;\bar{\omega}) = (-1)^m s(s+1)\cdots(s+m-1)\zeta_N(s+m,x;\bar{\omega}) \quad \text{for all } m\in\mathbb{Z}^+; \tag{2.5}$$

$$\zeta_N(-k,x;\bar{\omega}) = \frac{(-1)^N k!}{(N+k)!}B_{N,N+k}(x;\bar{\omega}) \quad \text{for all } k\in\mathbb{Z}_0, \tag{2.6}$$

where the $B_{N,n}(x;\bar{\omega})$ are the N-th order Bernoulli polynomials defined below.

The *Bernoulli polynomial* $B_{N,n}(x;\bar{\omega})$ *of order* N *and degree* n *with parameter vector* $\bar{\omega}=(\omega_1,\ldots,\omega_N)$ is defined by the exponential generating function

$$\frac{t^N e^{xt}}{(e^{\omega_1 t}-1)\cdots(e^{\omega_N t}-1)} = \sum_{n=0}^{\infty} B_{N,n}(x;\bar{\omega})\frac{t^n}{n!}. \tag{2.7}$$

When $N=0$ there is no parameter vector $\bar{\omega}$ but we still use the generating function to define $B_{0,n}(x;-)=x^n$. When $N=1$ and $\omega_1=1$ we recover the usual Bernoulli polynomials, that is, $B_{1,n}(x;1)=B_n(x)$ for all $n\in\mathbb{Z}_0$. If all $\omega_i=1$ then $B_n^{(N)}(x) := B_{N,n}(x;1,\ldots,1)$ are the polynomials studied in [Y]. The polynomials $C_n(x;\omega_1,\omega_2) := B_{2,n}(x;\omega_1,\omega_2)$ appear in [St]. The polynomial $\omega_1\cdots\omega_N B_{N,n}(x;\bar{\omega})$ is a homogeneous polynomial of degree n in $x,\omega_1,\ldots,\omega_N$. The following additional properties specialize to well-known properties of the usual Bernoulli polynomials and may be proved along similar lines:

$$B_{N,n}(x;\bar{\omega}) = \sum_{k=0}^{n}\binom{n}{k}B_{N,k}(0;\bar{\omega})x^{n-k}; \tag{2.8}$$

$$B_{N,n}(\omega_1+\cdots+\omega_N-x;\bar{\omega}) = (-1)^n B_{N,n}(x;\bar{\omega}); \tag{2.9}$$

$$\frac{d}{dx}B_{N,n+1}(x;\bar{\omega}) = (n+1)B_{N,n}(x;\bar{\omega}); \tag{2.10}$$

$$\sum_{j=0}^{M-1} B_{K,n}(x+j\omega_{K+1};\omega_1,\ldots,\omega_K) = \frac{B_{K+1,n+1}(x+M\omega_{K+1};\omega_1,\ldots,\omega_{K+1}) - B_{K+1,n+1}(x;\omega_1,\ldots,\omega_{K+1})}{n+1}. \tag{2.11}$$

The definition of the function $\log(\Gamma_N(x;\bar{\omega}))$ was already given in Section 1 by Eq. (1.2). The following properties follow readily from (2.3), (2.4), and (2.5), respectively:

$$\log(\Gamma_N(x+\omega_N;\omega_1,\ldots,\omega_N)) - \log(\Gamma_N(x;\omega_1,\ldots,\omega_N)) = -\log(\Gamma_{N-1}(x;\omega_1,\ldots,\omega_{N-1})); \tag{2.12}$$

$$\log(\Gamma_N(cx;c\bar{\omega})) = \log(\Gamma_N(x;\bar{\omega})) - \zeta_N(0,x;\bar{\omega})\log c \quad \text{for all } c\in\mathbb{R}^+; \tag{2.13}$$

$$\frac{\partial^{N+k}}{\partial x^{N+k}}\log(\Gamma_N(x;\bar{\omega})) = (-1)^{N+k}(N+k-1)!\,\zeta_N(N+k,x;\bar{\omega}), \tag{2.14}$$

where $k \in \mathbb{Z}^+$ (recall that $\zeta_N(m, x; \bar{\omega})$ is not defined when $m \in \{1, \ldots, N\}$).

For a prime number p we use $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ to denote the ring of p-adic integers, the field of p-adic numbers, and the completion of an algebraic closure of $\mathbb{Q}_p$, respectively. Let $|\cdot|_p$ denote the unique absolute value defined on $\mathbb{C}_p$ normalized by $|p|_p = p^{-1}$. Given $x \in \mathbb{C}_p^\times$, we let $\nu_p(x) \in \mathbb{Q}$ denote the unique exponent such that $|x|_p = p^{-\nu_p(x)}$. As usual, we set $\nu_p(0) = \infty$. For a vector $\bar{\omega} = (\omega_1, \ldots, \omega_N) \in \mathbb{C}_p^N$ we define its norm $||\bar{\omega}||_p$ by $||\bar{\omega}||_p = \max\{|\omega_1|_p, \ldots, |\omega_N|_p\}$.

We choose an embedding of $\overline{\mathbb{Q}}$ into $\mathbb{C}_p$ and fix it once and for all. Let $p^{\mathbb{Q}}$ denote the image in $\mathbb{C}_p^\times$ of the set of positive real rational powers of p under this embedding. Let μ denote the group of roots of unity in $\mathbb{C}_p^\times$ of order not divisible by p. If $x \in \mathbb{C}_p$, $|x|_p = 1$ then there is a unique element $\hat{x} \in \mu$ such that $|x - \hat{x}|_p < 1$ (called the *Teichmüller representative* of x); it may be defined by $\hat{x} = \lim_{n\to\infty} x^{p^{n!}}$. We extend this definition to $x \in \mathbb{C}_p^\times$ by

$$\hat{x} := \widehat{(x/p^{\nu_p(x)})}, \tag{2.15}$$

that is, we define $\hat{x} = \hat{u}$ if $x = p^r u$ with $p^r \in p^{\mathbb{Q}}$ and $|u|_p = 1$. We then define the function $\langle\cdot\rangle$ on $\mathbb{C}_p^\times$ by $\langle x\rangle = p^{-\nu_p(x)} x/\hat{x}$ (note that this definition disagrees with that of [W] when $p = 2$). This yields an internal direct product decomposition of multiplicative groups

$$\mathbb{C}_p^\times \simeq p^{\mathbb{Q}} \times \mu \times D \tag{2.16}$$

where $D = \{x \in \mathbb{C}_p : |x - 1| < 1\}$, given by

$$x = p^{\nu_p(x)} \cdot \hat{x} \cdot \langle x\rangle \mapsto (p^{\nu_p(x)}, \hat{x}, \langle x\rangle). \tag{2.17}$$

This decomposition of $\mathbb{C}_p^\times$ depends on our choice of embedding of $\overline{\mathbb{Q}}$ into $\mathbb{C}_p$; the projections $p^{\nu_p(x)}, \hat{x}, \langle x\rangle$ are only determined up to roots of unity. However for $x \in \mathbb{Q}_p^\times$ the projections $p^{\nu_p(x)}, \hat{x}, \langle x\rangle$ are uniquely determined and do not depend on the choice of embedding. It will be observed that the projections $x \mapsto p^{\nu_p(x)}$ and $x \mapsto \hat{x}$ are constant on discs of the form $\{x \in \mathbb{C}_p : |x - y|_p < |y|_p\}$ and therefore have derivative zero, whereas the projection $x \mapsto \langle x\rangle$ has derivative $\frac{d}{dx}\langle x\rangle = \langle x\rangle/x$.

For $x \in D$ the Iwasawa logarithm function $\log_p$ is defined by the usual power series

$$\log_p x = -\sum_{n=1}^{\infty} \frac{(1-x)^n}{n} \tag{2.18}$$

and is extended to a continuous function on $\mathbb{C}_p^\times$ by defining $\log_p x = \log_p \langle x\rangle$. We emphasize that this definition is independent of the choice made to define the $\langle\cdot\rangle$ function, because the Iwasawa logarithm is the unique continuous homomorphism extending $\log_p$ from D to $\mathbb{C}_p^\times$ which satisfies $\log_p p = 0$.

For $x \in \mathbb{C}_p^\times$ and $s \in \mathbb{C}_p$ we define $\langle x\rangle^s$ ([Sc], p. 141) by

$$\langle x\rangle^s = \sum_{n=0}^{\infty} \binom{s}{n} (\langle x\rangle - 1)^n \tag{2.19}$$

when this sum converges.

Proposition 2.1. *For any $x \in \mathbb{C}_p^\times$ the function $s \mapsto \langle x \rangle^s$ is a C^∞ function of s on $\mathbb{Z}_p$ and is analytic on a disc of positive radius about $s = 0$; on this disc it is locally analytic as a function of x and independent of the choice made to define the $\langle \cdot \rangle$ function. If x lies in a finite extension K of $\mathbb{Q}_p$ whose ramification index over $\mathbb{Q}_p$ is less than $p - 1$ then $s \mapsto \langle x \rangle^s$ is analytic for $|s|_p < |\pi|_p^{-1} p^{-1/(p-1)}$, where (π) is the maximal ideal of the ring of integers $\mathfrak{O}_K$ of K. If $s \in \mathbb{Z}_p$, the function $x \mapsto \langle x \rangle^s$ is an analytic function of x on any disc of the form $\{x \in \mathbb{C}_p : |x - y|_p < |y|_p\}$.*

Proof. That $x \mapsto \langle x \rangle^s$ is a locally analytic function of x as described follows from ([Sc], Theorem 47.8); that $s \mapsto \langle x \rangle^s$ is a C^∞ function of s on $\mathbb{Z}_p$ is given in ([Sc], Corollary 54.2).

The p-adic exponential function $\exp_p$ is defined for $|x|_p < p^{-1/(p-1)}$ by

$$\exp_p(x) = \sum_{n=0}^{\infty} x^n/n!. \tag{2.20}$$

Therefore for any given $x \in \mathbb{C}_p^\times$, the function

$$\exp_p(s \log_p \langle x \rangle) = \sum_{k=0}^{\infty} s^k (\log_p \langle x \rangle)^k / k! \tag{2.21}$$

is an analytic function of s for $|s|_p < p^{-1/(p-1)} |\log \langle x \rangle|_p^{-1}$. For such s we have

$$\begin{aligned} \exp_p(s \log_p \langle x \rangle) &= \sum_{k=0}^{\infty} s^k \frac{(\log_p \langle x \rangle)^k}{k!} \\ &= \sum_{k=0}^{\infty} s^k \left(\sum_{n=k}^{\infty} s(n,k) \frac{(\langle x \rangle - 1)^n}{n!} \right) \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \frac{s(n,k) s^k}{n!} \right) (\langle x \rangle - 1)^n \\ &= \sum_{n=0}^{\infty} \binom{s}{n} (\langle x \rangle - 1)^n = \langle x \rangle^s, \end{aligned} \tag{2.22}$$

showing that $s \mapsto \langle x \rangle^s$ is analytic on this disc; here the *Stirling numbers of the first kind* $s(n,k)$ are the integers defined by

$$n! \binom{x}{n} = x(x-1) \cdots (x-n-1) = \sum_{k=0}^{n} s(n,k) x^k \tag{2.23}$$

or, equivalently, by the generating function

$$\frac{(\log(1+x))^k}{k!} = \sum_{n=k}^{\infty} s(n,k) \frac{x^n}{n!}. \tag{2.24}$$

Since the p-adic logarithm is independent of the choice made to define the $\langle\cdot\rangle$ function and we have $\langle x\rangle^s = \exp_p(s\log_p\langle x\rangle)$ on this disc, the function $s \mapsto \langle x\rangle^s$ is independent of the choice of embedding on this disc.

For $x \in K$ as described we have $\langle x\rangle - 1 \in (\pi)$; thus if the ramification index of K over $\mathbb{Q}_p$ is less than $p-1$ then $|\langle x\rangle - 1|_p < p^{-1/(p-1)}$ and therefore $|\log_p\langle x\rangle|_p = |\langle x\rangle - 1|_p$ ([W], Lemma 5.5); this proves that $s \mapsto \langle x\rangle^s$ is analytic for $|s|_p < |\pi|_p^{-1}p^{-1/(p-1)}$ for $x \in K$ as given. □

Let $f : \mathbb{Z}_p \to \mathbb{C}_p$ be strictly differentiable on $\mathbb{Z}_p$ (see [C], §11.1.2) and denote this by $f \in S^1(\mathbb{Z}_p)$. The *Volkenborn integral* of f is defined by

$$\int_{\mathbb{Z}_p} f(t)\,dt = \lim_{k\to\infty} p^{-k}\sum_{j=0}^{p^k-1} f(j) \tag{2.25}$$

and this limit always exists when $f \in S^1(\mathbb{Z}_p)$ ([Sc], §55). For such functions we have:

$$\int_{\mathbb{Z}_p} f(t+1)\,dt = \int_{\mathbb{Z}_p} f(t)\,dt + f'(0); \tag{2.26}$$

$$\int_{\mathbb{Z}_p} f(t+1)\,dt = \int_{\mathbb{Z}_p} f(-t)\,dt; \tag{2.27}$$

$$\int_{\mathbb{Z}_p} f(t)\,dt = \frac{1}{m}\sum_{j=0}^{m-1}\int_{\mathbb{Z}_p} f(j+mt)\,dt \quad \text{for } m \in \mathbb{Z}^+. \tag{2.28}$$

The multiple Volkenborn integrals we consider are all computable as iterated integrals. At the kth iteration, with $1 \le k \le N$, we integrate

$$\int_{\mathbb{Z}_p} F_k(t_k, t_{k+1}, \ldots, t_N)\,dt_k,$$

given that $F_k(t_k, t_{k+1}, \ldots, t_N) \in S^1(\mathbb{Z}_p)$ for each fixed vector $(t_{k+1}, \ldots, t_N) \in \mathbb{Z}_p^{N-k}$. Under these conditions, we use the notation

$$\int_{\mathbb{Z}_p^N} f(\bar{t})\,d\bar{t}, \qquad \text{where } \bar{t} = (t_1, \ldots, t_N), \tag{2.29}$$

to denote the N-fold iterated Volkenborn integral

$$\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} f(t_1, \ldots, t_N)\,dt_1\cdots dt_N. \tag{2.30}$$

Lemma 2.2. *For any $x \in \mathbb{C}_p$ and $\omega_1, \ldots, \omega_N \in \mathbb{C}_p^\times$, we have for $1 \le k \le N$*

$$\int_{\mathbb{Z}_p} B_{k-1,n}(x + \omega_k t_k + \cdots + \omega_N t_N; \omega_1, \ldots, \omega_{k-1})\,dt_k$$
$$= \omega_k B_{k,n}(x + \omega_{k+1}t_{k+1} + \cdots + \omega_N t_N; \omega_1, \ldots, \omega_k),$$

where $B_{0,n}(x; -) = x^n$, and $\omega_{k+1}t_{k+1} + \cdots + \omega_N t_N = 0$ when $k = N$.

Proof. Using (2.11) and (2.10), we have

$$\begin{aligned}
&\int_{\mathbb{Z}_p} B_{k-1,n}(x+\omega_k t_k+\cdots+\omega_N t_N;\omega_1,\ldots,\omega_{k-1})\,dt_k \\
&=\lim_{l\to\infty} p^{-l}\sum_{t_k=0}^{p^l-1} B_{k-1,n}(x+\omega_k t_k+\cdots+\omega_N t_N;\omega_1,\ldots,\omega_{k-1}) \\
&=\lim_{l\to\infty}\frac{p^{-l}}{n+1}(B_{k,n+1}(x+\omega_{k+1}t_{k+1}+\cdots+\omega_N t_N+p^l\omega_k;\omega_1,\ldots,\omega_k) \\
&\qquad -B_{k,n+1}(x+\omega_{k+1}t_{k+1}+\cdots+\omega_N t_N;\omega_1,\ldots,\omega_k)) \\
&=\frac{\omega_k}{n+1}B'_{k,n+1}(x+\omega_{k+1}t_{k+1}+\cdots+\omega_N t_N;\omega_1,\ldots,\omega_k) \\
&=\omega_k B_{k,n}(x+\omega_{k+1}t_{k+1}+\cdots+\omega_N t_N;\omega_1,\ldots,\omega_k). \qquad \square
\end{aligned}$$

Corollary 2.3. *For any $x\in\mathbb{C}_p$ and $\omega_1,\ldots,\omega_N\in\mathbb{C}_p^\times$, we have*

$$\int_{\mathbb{Z}_p^N}(x+\bar{\omega}\cdot\bar{t})^n\,d\bar{t}=\omega_1\cdots\omega_N B_{N,n}(x;\bar{\omega})$$

for all nonnegative integers N and n.

Proof. When $N=0$, there is no integration and this reduces back to the definition $x^n=B_{0,n}(x;-)$. When $N\geq 1$, we apply Lemma 2.2 inductively for $k=1,\ldots,N$. $\square$

Remark. One may regard this corollary as providing a p-adic definition of the order N Bernoulli polynomials.

3. *p*-adic multiple zeta and log gamma functions

Suppose that $\omega_1,\ldots,\omega_N\in\mathbb{C}_p^\times$, and let Λ denote the $\mathbb{Z}_p$-linear span of $\{\omega_1,\ldots,\omega_N\}$. For $x\in\mathbb{C}_p\setminus\Lambda$, we define the p-adic multiple zeta function $\zeta_{p,N}(s,x;\bar{\omega})$ by

$$\begin{aligned}
\zeta_{p,N}(s,x;\bar{\omega})&=\frac{1}{\omega_1\cdots\omega_N(s-1)\cdots(s-N)} \\
&\quad\times\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\frac{(x+\omega_1t_1+\cdots+\omega_Nt_N)^N}{\langle x+\omega_1t_1+\cdots+\omega_Nt_N\rangle^s}dt_1\cdots dt_N \qquad (3.1)
\end{aligned}$$

which we write as

$$\zeta_{p,N}(s,x;\bar{\omega})=\frac{1}{\omega_1\cdots\omega_N(s-1)\cdots(s-N)}\int_{\mathbb{Z}_p^N}\frac{(x+\bar{\omega}\cdot\bar{t})^N}{\langle x+\bar{\omega}\cdot\bar{t}\rangle^s}\,d\bar{t}. \qquad (3.2)$$

Observe that $\zeta_{p,1}(s,x;1)$ is defined in a similar way to the function $\zeta_p(s,x)$ defined by Cohen ([C], p. 283) but with a slight difference in the integrand. The function $\zeta_{p,1}(s,x;1)$ still nicely interpolates the complex-valued Hurwitz zeta function (see Theorem 3.2(v)).

Theorem 3.1. *For any choices of* $\omega_1, \ldots, \omega_N, x \in \mathbb{C}_p^\times$ *with* $x \notin \Lambda$ *the function* $\zeta_{p,N}(s, x; \bar{\omega})$ *is a* C^∞ *function of* s *on* $\mathbb{Z}_p \setminus \{1, \ldots, N\}$*, and is an analytic function of* s *on a disc of positive radius about* $s = 0$*; on this disc it is locally analytic as a function of* x *and independent of the choice made to define the* $\langle \cdot \rangle$ *function. If* $\omega_1, \ldots, \omega_N, x$ *are so chosen to lie in a finite extension* K *of* $\mathbb{Q}_p$ *whose ramification index over* $\mathbb{Q}_p$ *is less than* $p - 1$ *then* $\zeta_{p,N}(s, x; \bar{\omega})$ *is analytic for* $s \in \mathbb{C}_p$ *such that* $|s|_p < |\pi|_p^{-1} p^{-1/(p-1)}$*, except for simple poles at* $s = 1, \ldots, N$*. If* $s \in \mathbb{Z}_p \setminus \{1, \ldots, N\}$*, the function* $\zeta_{p,N}(s, x; \bar{\omega})$ *is locally analytic as a function of* x *on* $\mathbb{C}_p \setminus \Lambda$*.*

Proof. This follows from Proposition 2.1 and the definition of $\zeta_{p,N}$. Since $\mathbb{Z}_p^N$ is compact, for each $x \in \mathbb{C}_p \setminus \Lambda$ we may choose $\rho > 0$ so that $\langle x + \bar{\omega} \cdot \bar{t} \rangle^s$ is analytic for $|s|_p < \rho$ for all $t \in \mathbb{Z}_p^N$ and also $\rho < \min\{|1|_p, \ldots, |N|_p\}$; then $\zeta_{p,N}(s, x; \bar{\omega})$ is analytic for $|s|_p < \rho$. □

Remark. For arbitrary $\bar{\omega}$ and x this zeta function is not necessarily meromorphic on a disc containing $\mathbb{Z}_p$, but is always analytic on some disc about $s = 0$. However, for the applications we have in mind the ω_i and x lie in $\mathbb{Q}_p$ so $\zeta_{p,N}$ will be meromorphic on a disc containing $\mathbb{Z}_p$.

Theorem 3.2. *The function* $\zeta_{p,N}(s, x; \bar{\omega})$ *has the following properties:*

(i). *For all* $x \in \mathbb{C}_p \setminus \Lambda$ *the function* $\zeta_{p,N}(s, x; \bar{\omega})$ *satisfies the difference equation*

$$\begin{aligned} \zeta_{p,N}(s, x + \omega_N; \omega_1, \ldots, \omega_N) - \zeta_{p,N}(s, x; \omega_1, \ldots, \omega_N) \\ = -\zeta_{p,N-1}(s, x; \omega_1, \ldots, \omega_{N-1}), \end{aligned}$$

with the convention $\zeta_{p,0}(s, x; -) = \langle x \rangle^{-s}$.

(ii). *For all* $c \in \mathbb{C}_p^\times$ *and all* $x \in \mathbb{C}_p \setminus \Lambda$ *we have*

$$\zeta_{p,N}(s, cx; c\bar{\omega}) = \langle c \rangle^{-s} \zeta_{p,N}(s, x; \bar{\omega}).$$

(iii). *For all* $x \in \mathbb{C}_p \setminus \Lambda$ *we have the reflection functional equation*

$$\zeta_{p,N}(s, \omega_1 + \cdots + \omega_N - x; \bar{\omega}) = (-1)^N \zeta_{p,N}(s, x; \bar{\omega}).$$

For $p \neq 2$ *this holds for all* s *where* $\zeta_{p,N}$ *is defined; for* $p = 2$ *it does not hold for all* s *but does hold on a disc about* $s = 0$.

(iv). *For all* $x \in \mathbb{C}_p \setminus \Lambda$ *and all positive integers* m *we have the multiplication formula (distribution relation)*

$$\zeta_{p,N}(s, x; \bar{\omega}) = \langle m \rangle^{-s} \sum_{0 \leq j_i < m} \zeta_{p,N}\left(s, \frac{x + \bar{\jmath} \cdot \bar{\omega}}{m}; \bar{\omega}\right)$$

where the sum is over all vectors $\bar{\jmath} = (j_1, \ldots, j_N)$ *with* $0 \leq j_i < m$*. In particular for any positive integer* k *we have*

$$\zeta_{p,N}(s, x; \bar{\omega}) = \sum_{0 \leq j_i < p^k} \zeta_{p,N}\left(s, \frac{x + \bar{\jmath} \cdot \bar{\omega}}{p^k}; \bar{\omega}\right).$$

(v). Suppose that $\omega_1, \ldots, \omega_N \in \overline{\mathbb{Q}}$ are positive real numbers and $x \in \overline{\mathbb{Q}}$ is a complex number with positive real part. Under our fixed embedding of $\overline{\mathbb{Q}}$ into $\mathbb{C}_p$, suppose that $|x|_p > ||\bar{\omega}||_p$. Then for all nonnegative integers k,

$$\zeta_{p,N}(-k, x; \bar{\omega}) = \left(\frac{\langle x\rangle}{x}\right)^k \zeta_N(-k, x; \bar{\omega}).$$

(vi). Suppose $\omega_1, \ldots, \omega_N, x \in \mathbb{C}_p^\times$ and $|x|_p > ||\bar{\omega}||_p$. Then for any positive integer m the identity

$$\frac{\partial^m}{\partial x^m}\zeta_{p,N}(s, x; \bar{\omega}) = (-1)^m \left(\frac{\langle x\rangle}{x}\right)^m s(s+1)\cdots(s+m-1)\zeta_{p,N}(s+m, x; \bar{\omega})$$

holds if $s \in \mathbb{Z}_p \setminus \{1-m, \ldots, N\}$; this identity also holds for $|s|_p < |\pi|_p^{-1}p^{-1/(p-1)}$ with $s \notin \{1-m, ..., N\}$ if x and all ω_i lie in a finite extension K of $\mathbb{Q}_p$ whose ring of integers has maximal ideal (π) with ramification index over $\mathbb{Q}_p$ less than $p-1$.

Remarks. Properties (i), (ii), and (iv) are direct analogues of the classical difference equation, homogeneity property, and distribution relation of the Barnes multiple zeta function (see (2.3), (2.4), and p. 844 of [KK], respectively). The reflection formula (iii) has no analogue in the complex case except at nonpositive integer values of s. Property (vi) corresponds directly to (2.5). The exact twist needed to connect the p-adic and complex multiple zeta function values at the nonpositive integers is given in part (v). Under appropriate conditions, we may combine properties (vi), (v), and Eq. (2.5) to prove that

$$\frac{\partial^m}{\partial x^m}\zeta_{p,N}(-k, x; \bar{\omega}) = \left(\frac{\langle x\rangle}{x}\right)^k \frac{\partial^m}{\partial x^m}\zeta_N(-k, x; \bar{\omega})$$

when k is a nonnegative integer. We also remark that the p-adic multiple zeta function defined by Kashio [K] does not satisfy (i) when $N = 1$; satisfies (ii) only for $|c|_p = 1$ (see [KY], (2.8)); and satisfies (v) only under the additional assumption that $|x|_p \geq 1$.

Proof. Part (ii) is immediate from the definition. For (i), define a function $f(t_N)$ on $\mathbb{Z}_p$ by the $(N-1)$-fold integral

$$f(t_N) = \frac{1}{\omega_1\cdots\omega_N(s-1)\cdots(s-N)} \times \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} \frac{(x+\omega_1 t_1+\cdots+\omega_N t_N)^N}{\langle x+\omega_1 t_1+\cdots+\omega_N t_N\rangle^s} dt_1\cdots dt_{N-1}. \quad (3.3)$$

Then we have on the one hand

$$\zeta_{p,N}(s, x+\omega_N; \omega_1, \ldots, \omega_N) - \zeta_{p,N}(s, x; \omega_1, \ldots, \omega_N) = \int_{\mathbb{Z}_p} (f(t_N+1) - f(t_N))\, dt_N \quad (3.4)$$

and on the other hand

$$f'(0) = -\zeta_{p,N-1}(s, x; \omega_1, \ldots, \omega_{N-1}), \tag{3.5}$$

which proves (i), since $\int_{\mathbb{Z}_p} (f(t_N + 1) - f(t_N))\, dt_N = f'(0)$.

For (iii) and (iv) we define

$$f(x, \bar{t}) = \frac{(x + \bar{\omega} \cdot \bar{t})^N}{\omega_1 \cdots \omega_N (s-1) \cdots (s-N) \langle x + \bar{\omega} \cdot \bar{t} \rangle^s}. \tag{3.6}$$

Then we have

$$\begin{aligned}
&\zeta_{p,N}(s, \omega_1 + \cdots + \omega_N - x; \bar{\omega}) \\
&= \int_{\mathbb{Z}_p^N} f(\omega_1 + \cdots + \omega_N - x, \bar{t})\, d\bar{t} \\
&= \int_{\mathbb{Z}_p^N} f(\omega_1 + \cdots + \omega_{N-1} - x, t_1, \ldots, t_{N-1}, t_N + 1)\, d\bar{t} \\
&= \int_{\mathbb{Z}_p^N} f(\omega_1 + \cdots + \omega_{N-1} - x, t_1, \ldots, t_{N-1}, -t_N)\, d\bar{t} \\
&= \cdots \\
&= \int_{\mathbb{Z}_p^N} f(-x, -t_1, \ldots, -t_N)\, d\bar{t} \\
&= (-1)^N \int_{\mathbb{Z}_p^N} f(x, \bar{t})\, d\bar{t} = (-1)^N \zeta_{p,N}(s, x; \bar{\omega}),
\end{aligned} \tag{3.7}$$

giving (iii). The validity of the penultimate equality depends on the fact that $\langle -z \rangle^s = \langle z \rangle^s$; when $p \neq 2$ this is valid since $\langle -z \rangle = \langle z \rangle$. For $p = 2$ we have $\langle -z \rangle = -\langle z \rangle$ so the equation is not valid for general s, but there is a disc about $s = 0$ on which $\langle -z \rangle^s = \exp_p(s \log_p(-\langle z \rangle)) = \exp_p(s \log_p \langle z \rangle) = \langle z \rangle^s$, so on this disc the equation is valid. We may use (3.6) and (2.28) to compute

$$\begin{aligned}
\zeta_{p,N}(s, x; \bar{\omega}) &= \int_{\mathbb{Z}_p^N} f(x, \bar{t})\, d\bar{t} \\
&= m^{-N} \sum_{0 \le j_i < m} \int_{\mathbb{Z}_p^N} f(x, \bar{\jmath} + m\bar{t})\, d\bar{t} \qquad (3.8) \\
&= m^{-N} \sum_{0 \le j_i < m} \int_{\mathbb{Z}_p^N} f(x + \bar{\jmath} \cdot \bar{\omega}, m\bar{t})\, d\bar{t} \\
&= \langle m \rangle^{-s} \sum_{0 \le j_i < m} \int_{\mathbb{Z}_p^N} f\left(\frac{x + \bar{\jmath} \cdot \bar{\omega}}{m}, \bar{t} \right) d\bar{t} \\
&= \langle m \rangle^{-s} \sum_{0 \le j_i < m} \zeta_{p,N}\left(s, \frac{x + \bar{\jmath} \cdot \bar{\omega}}{m}; \bar{\omega} \right),
\end{aligned}$$

proving (iv).

Note that that if $|x|_p > ||\bar{\omega}||_p$, then $\langle x + \bar{\omega} \cdot \bar{t} \rangle = \frac{\langle x \rangle}{x}(x + \bar{\omega} \cdot \bar{t})$ for all $\bar{t} \in \mathbb{Z}_p^N$, and this observation is used in proving both (v) and (vi). For given x and $\bar{\omega}$ satisfying these hypotheses we have $\frac{\partial}{\partial x}\langle x + \bar{\omega} \cdot \bar{t} \rangle^s = s \langle x + \bar{\omega} \cdot \bar{t} \rangle^s (x + \bar{\omega} \cdot \bar{t})^{-1}$ uniformly for $\bar{t} \in \mathbb{Z}_p^N$. Differentiating using the definition of $\zeta_{p,N}$ yields

$$\frac{\partial}{\partial x} \zeta_{p,N}(s, x; \bar{\omega}) = -s \frac{\langle x \rangle}{x} \zeta_{p,N}(s + 1, x; \bar{\omega}) \tag{3.9}$$

proving (vi) for $m = 1$. The general statement of (vi) follows by induction on m. Similarly, by definition and the use of Corollary 2.3 and Eq. (2.6) we have

$$\begin{aligned}
\zeta_{p,N}(-k, x; \bar{\omega}) &= \frac{(-1)^N}{\omega_1 \cdots \omega_N (k+1) \cdots (k+N)} \int_{\mathbb{Z}_p^N} (x + \bar{\omega} \cdot \bar{t})^N \langle x + \bar{\omega} \cdot \bar{t} \rangle^k \, d\bar{t} \\
&= \frac{(-1)^N x^{-k} \langle x \rangle^k}{\omega_1 \cdots \omega_N (k+1) \cdots (k+N)} \int_{\mathbb{Z}_p^N} (x + \bar{\omega} \cdot \bar{t})^{N+k} \, d\bar{t} \\
&= \frac{(-1)^N x^{-k} \langle x \rangle^k}{\omega_1 \cdots \omega_N (k+1) \cdots (k+N)} \int_{\mathbb{Z}_p^N} B_{0,N+k}(x + \bar{\omega} \cdot \bar{t}; -) \, d\bar{t} \\
&= \frac{(-1)^N x^{-k} \langle x \rangle^k}{(k+1) \cdots (k+N)} B_{N,N+k}(x; \bar{\omega}) \qquad (3.10) \\
&= \left(\frac{\langle x \rangle}{x} \right)^k \zeta_N(-k, x; \bar{\omega}),
\end{aligned}$$

which proves (v). $\square$

Given $\omega_1, \ldots, \omega_N \in \mathbb{C}_p^\times$ and $x \in \mathbb{C}_p \setminus \Lambda$, we define in direct analogy to Eq. (1.2), the p-adic multiple log gamma function $G_{p,N}(x; \bar{\omega})$ by

$$G_{p,N}(x; \bar{\omega}) = \frac{\partial}{\partial s} \zeta_{p,N}(s, x; \bar{\omega}) \bigg|_{s=0}. \tag{3.11}$$

Therefore we have

$$\begin{aligned}
G_{p,N}(x; \bar{\omega}) &= \frac{(-1)^{N+1}}{\omega_1 \cdots \omega_N N!} \\
\times \int_{\mathbb{Z}_p^N} &(x + \bar{\omega} \cdot \bar{t})^N \left[\log_p(x + \bar{\omega} \cdot \bar{t}) - \left(1 + \frac{1}{2} + \cdots + \frac{1}{N} \right) \right] d\bar{t}. \quad (3.12)
\end{aligned}$$

When $N = 1$ and $\omega_1 = 1$, the function $G_{p,1}(x; 1)$ defined by (3.12) matches precisely the Volkenborn integral definition of $G_p(x)$ due to Diamond ([D], p. 326).

Theorem 3.3. *For any $\omega_1, \ldots, \omega_N \in \mathbb{C}_p^\times$ the function $G_{p,N}(x; \bar{\omega})$ is independent of the choice made to define the $\langle \cdot \rangle$ function, and is locally analytic as a function of x on $\mathbb{C}_p \setminus \Lambda$.*

PROOF. Although $\zeta_{p,N}(s,x;\bar{\omega})$ is defined in terms of the $\langle\cdot\rangle$ function, which depends on a choice of embedding of $\overline{\mathbb{Q}}$ into $\mathbb{C}_p$, Eq. (3.12) shows that $G_{p,N}$ may be defined without reference to $\langle\cdot\rangle$. The local analyticity of $G_{p,N}$ is immediate from the local analyticity of $\zeta_{p,N}(x,s;\bar{\omega})$. $\square$

Theorem 3.4. *The function $G_{p,N}(x;\bar{\omega})$ has the following properties:*

(i). For all $x \in \mathbb{C}_p \setminus \Lambda$ the function $G_{p,N}(x;\bar{\omega})$ satisfies the difference equation

$$G_{p,N}(x+\omega_N;\omega_1,\ldots,\omega_N) - G_{p,N}(x;\omega_1,\ldots,\omega_N) = -G_{p,N-1}(x;\omega_1,\ldots,\omega_{N-1}),$$

with the convention $G_{p,0}(x;-) = -\log_p x$.

(ii). For all $c \in \mathbb{C}_p^\times$ and all $x \in \mathbb{C}_p \setminus \Lambda$ we have

$$G_{p,N}(cx;c\bar{\omega}) = G_{p,N}(x;\bar{\omega}) - \zeta_{p,N}(0,x;\bar{\omega})\log_p c.$$

(iii). For all $x \in \mathbb{C}_p \setminus \Lambda$ we have the reflection functional equation

$$G_{p,N}(\omega_1+\cdots+\omega_N - x;\bar{\omega}) = (-1)^N G_{p,N}(x;\bar{\omega}).$$

(iv). For all $x \in \mathbb{C}_p \setminus \Lambda$ and all positive integers m we have the multiplication formula (distribution relation)

$$G_{p,N}(x;\bar{\omega}) = -\zeta_{p,N}(0,x;\bar{\omega})\log_p m + \sum_{0\le j_i<m} G_{p,N}\left(\frac{x+\bar{\jmath}\cdot\bar{\omega}}{m};\bar{\omega}\right)$$

where the sum is over all vectors $\bar{\jmath} = (j_1,\ldots,j_N)$ with $0 \le j_i < m$. In particular for any positive integer k we have

$$G_{p,N}(x;\bar{\omega}) = \sum_{0\le j_i<p^k} G_{p,N}\left(\frac{x+\bar{\jmath}\cdot\bar{\omega}}{p^k};\bar{\omega}\right).$$

(v). For all $x \in \mathbb{C}_p \setminus \Lambda$ we have

$$\frac{\partial^k}{\partial x^k}G_{p,N}(x;\bar{\omega}) = \frac{(-1)^k}{\omega_{N-k+1}\cdots\omega_N}\int_{\mathbb{Z}_p^k} G_{p,N-k}(x_k;\bar{\omega}_k)\,dt_{N-k+1}\cdots dt_N$$

for $1 \le k \le N$, where $x_k = x + \omega_{N-k+1}t_{N-k+1} + \cdots + \omega_N t_N$ and $\bar{\omega}_k = (\omega_1,\ldots,\omega_{N-k})$; in particular

$$\frac{\partial^N}{\partial x^N}G_{p,N}(x;\bar{\omega}) = \frac{(-1)^{N+1}}{\omega_1\cdots\omega_N}\int_{\mathbb{Z}_p^N}\log_p(x+\bar{\omega}\cdot\bar{t})\,d\bar{t}.$$

Furthermore, for all positive integers k we have

$$\frac{\partial^{N+k}}{\partial x^{N+k}}G_{p,N}(x;\bar{\omega}) = \frac{(-1)^{N+k}(k-1)!}{\omega_1\cdots\omega_N}\int_{\mathbb{Z}_p^N}\frac{d\bar{t}}{(x+\bar{\omega}\cdot\bar{t})^k};$$

and for $|x|_p > ||\bar{\omega}||_p$ we may write this as

$$\frac{\partial^{N+k}}{\partial x^{N+k}}G_{p,N}(x;\bar{\omega}) = (-1)^{N+k}(N+k-1)!\left(\frac{\langle x\rangle}{x}\right)^{N+k}\zeta_{p,N}(N+k,x;\bar{\omega}).$$

Remarks. Properties (i) and (ii) are direct analogues of (2.12) and (2.13), respectively. Property (iv) corresponds directly to the distribution relation in the complex case (see §41–42 in [B]). Property (iii) generalizes the reflection formula for Diamond's function G_p but has no analogue for the corresponding complex-valued functions. The final statement in (v) is an analogue to Eq. (2.14). The p-adic multiple log gamma function of Kashio [K] satisfies (i) only for $N > 1$ and satisfies (ii) only for $|c|_p = 1$.

Proof. Parts (i)-(iv) are obtained from parts (i)-(iv) of Theorem 3.2 by evaluating the partial derivative with respect to s at $s = 0$. For (v), observe that for given x and $\bar{\omega}$,

$$\frac{\partial}{\partial x}(x+\bar{\omega}\cdot\bar{t})^N\left[\log_p(x+\bar{\omega}\cdot\bar{t})-\left(1+\frac{1}{2}+\cdots+\frac{1}{N}\right)\right]$$
$$= N(x+\bar{\omega}\cdot\bar{t})^{N-1}\left[\log_p(x+\bar{\omega}\cdot\bar{t})-\left(1+\frac{1}{2}+\cdots+\frac{1}{N-1}\right)\right]$$

uniformly for $\bar{t}\in\mathbb{Z}_p^N$. Therefore

$$\frac{\partial}{\partial x}G_{p,N}(x;\bar{\omega}) = \frac{-1}{\omega_N}\int_{\mathbb{Z}_p} G_{p,N-1}(x+\omega_N t_N;\omega_1,\ldots,\omega_{N-1})\,dt_N, \tag{3.13}$$

giving the result for $k = 1$. The result for $1 \le k \le N$ follows by induction, using the convention $G_{p,0}(x;-) = -\log_p x$. The penultimate statement then follows from the $k = N$ case above and the observation that

$$\frac{\partial^k}{\partial x^k}\log_p(x+\bar{\omega}\cdot\bar{t}) = (-1)^{k-1}(k-1)!(x+\bar{\omega}\cdot\bar{t})^{-k}; \tag{3.14}$$

when $|x|_p > ||\bar{\omega}||_p$ we have $\langle x+\bar{\omega}\cdot\bar{t}\rangle = \frac{\langle x\rangle}{x}(x+\bar{\omega}\cdot\bar{t})$ for all $\bar{t}\in\mathbb{Z}_p^N$ and therefore from the definition

$$\zeta_{p,N}(N+k,x;\bar{\omega}) = \frac{1}{\omega_1\cdots\omega_N(N+k-1)\cdots k}\left(\frac{x}{\langle x\rangle}\right)^{N+k}\int_{\mathbb{Z}_p^N}\frac{d\bar{t}}{(x+\bar{\omega}\cdot\bar{t})^k},$$

we obtain the final statement. □

When $x \in \Lambda$ our functions $\zeta_{p,N}(s,x;\bar{\omega})$ and $G_{p,N}(x;\bar{\omega})$ are undefined. We remark that if $x \in \Lambda$ and $||\bar{\omega}||_p < 1$ then the functions defined by Kashio [K], which we will here denote by $\zeta^*_{p,N}$ and $L\Gamma_{p,N}$, are defined but always have the value zero. In the remaining case, when $||\bar{\omega}||_p \ge 1$ and $x \in \Lambda$ the functions of Kashio may be expressed in terms of our functions, as we now show.

Theorem 3.5. *Suppose that* $||\bar{\omega}||_p \ge 1$ *and* $x \in \Lambda$. *Then*

$$\zeta^*_{p,N}(s,x;\bar{\omega}) = \sum_{\substack{0\le j_i<p \\ |x+\bar{\jmath}\cdot\bar{\omega}|_p=||\bar{\omega}||_p}} \zeta_{p,N}\left(s,\frac{x+\bar{\jmath}\cdot\bar{\omega}}{p};\bar{\omega}\right)$$

and

$$L\Gamma_{p,N}(x;\bar\omega) = \sum_{\substack{0\le j_i<p\\ |x+\bar\jmath\cdot\bar\omega|_p=||\bar\omega||_p}} G_{p,N}\left(\frac{x+\bar\jmath\cdot\bar\omega}{p};\bar\omega\right)$$

where the sums are all over all vectors $\bar\jmath = (j_1,\ldots,j_N)$ *with* $0 \le j_i < p$ *and* $|x+\bar\jmath\cdot\bar\omega|_p = ||\bar\omega||_p$.

PROOF. Let us assume that $||\bar\omega||_p = 1$ and $x \in \Lambda$. In this case the zeta function of Kashio (see pp. 1639–40 of [KY]) may be defined as

$$\zeta^*_{p,N}(s,x;\bar\omega) = \int_{\mathbb{Z}_p^N} f^*(x,\bar t)\,d\bar t \tag{3.15}$$

where

$$f^*(x,\bar t) = \begin{cases} f(x,\bar t), & \text{if } |x+\bar\omega\cdot\bar t|_p = 1,\\ 0, & \text{if } |x+\bar\omega\cdot\bar t|_p < 1,\end{cases} \tag{3.16}$$

and $f(x,\bar t)$ is as defined in Eq. (3.6). By (2.28) and (3.16) we have

$$\begin{aligned}
\zeta^*_{p,N}(s,x;\bar\omega) &= \int_{\mathbb{Z}_p^N} f^*(x,\bar t)\,d\bar t\\
&= p^{-N}\sum_{0\le j_i<p}\int_{\mathbb{Z}_p^N} f^*(x,\bar\jmath+p\bar t)\,d\bar t\\
&= p^{-N}\sum_{0\le j_i<p}\int_{\mathbb{Z}_p^N} f^*(x+\bar\jmath\cdot\bar\omega,p\bar t)\,d\bar t\\
&= p^{-N}\sum_{\substack{0\le j_i<p\\ |x+\bar\jmath\cdot\bar\omega|_p=1}}\int_{\mathbb{Z}_p^N} f(x+\bar\jmath\cdot\bar\omega,p\bar t)\,d\bar t\\
&= \sum_{\substack{0\le j_i<p\\ |x+\bar\jmath\cdot\bar\omega|_p=1}}\int_{\mathbb{Z}_p^N} f\left(\frac{x+\bar\jmath\cdot\bar\omega}{p},\bar t\right)d\bar t \qquad (3.17)\\
&= \sum_{\substack{0\le j_i<p\\ |x+\bar\jmath\cdot\bar\omega|_p=||\bar\omega||_p}} \zeta_{p,N}\left(s,\frac{x+\bar\jmath\cdot\bar\omega}{p};\bar\omega\right),
\end{aligned}$$

which proves the first statement in the case $||\bar\omega||_p = 1$. The case of general $||\bar\omega||_p \ge 1$ may be deduced from the dilation relations (Theorem 3.2(ii) above and [K], Eq. (5.7)) of $\zeta_{p,N}$ and $\zeta^*_{p,N}$. The second statement is the partial derivative with respect to s at $s = 0$ of the first. □

Remark. Since $L\Gamma_{p,1}(x;1) = \log_p \Gamma_p(x)$ for $x \in \mathbb{Z}_p$ ([K], Lemma 5.5), when $N = 1$ and $\omega_1 = 1$ the second statement in Theorem 3.5 becomes the well-known relation ([L], p. 395)

$$\log_p \Gamma_p(x) = \sum_{\substack{0 \le j < p \\ |x+j|_p = 1}} G_p\left(\frac{x+j}{p}\right) \qquad (x \in \mathbb{Z}_p) \tag{3.18}$$

between the Morita p-adic gamma function $\Gamma_p(x)$ and the Diamond function $G_p(x)$.

4. Expansions for large x

In this section we give computationally efficient formulas for our functions in the case where the argument x has p-adic absolute value larger than the norm of $\bar{\omega}$. We remark that under this hypothesis, our functions agree exactly with the functions of Kashio [K] if $|x|_p \ge 1$, whereas if $|x|_p < 1$ then the functions of Kashio have the value zero.

Theorem 4.1. *Suppose $x, \omega_1, \ldots, \omega_N \in \mathbb{C}_p^\times$ and $|x|_p > ||\bar{\omega}||_p$. Then there is an identity of analytic functions*

$$\zeta_{p,N}(s, x; \bar{\omega}) = \frac{x^N \langle x \rangle^{-s}}{(s-1)\cdots(s-N)} \sum_{j=0}^{\infty} \binom{N-s}{j} B_{N,j}(0; \bar{\omega}) x^{-j}$$

on a disc of positive radius about $s = 0$. If in addition $\omega_1, \ldots, \omega_N, x$ are so chosen to lie in a finite extension K of $\mathbb{Q}_p$ whose ramification index over $\mathbb{Q}_p$ is less than $p-1$, then this formula is valid for $s \in \mathbb{C}_p \setminus \{1, \ldots, N\}$ such that $|s|_p < |\pi|_p^{-1} p^{-1/(p-1)}$, where (π) is the maximal ideal of the ring of integers $\mathfrak{O}_K$ of K.

Proof. Under the stated hypotheses, for all $\bar{t} \in \mathbb{Z}_p^N$, we use Proposition 2.1 to write

$$\begin{aligned}(x + \bar{\omega}\cdot\bar{t})^N \langle x + \bar{\omega}\cdot\bar{t}\rangle^{-s} &= x^N \langle x \rangle^{-s} \left(1 + \frac{\bar{\omega}\cdot\bar{t}}{x}\right)^{N-s} \\ &= x^N \langle x \rangle^{-s} \sum_{j=0}^{\infty} \binom{N-s}{j} (\bar{\omega}\cdot\bar{t})^j x^{-j}. \end{aligned} \tag{4.1}$$

as an identity of analyic functions. Writing $(\bar{\omega}\cdot\bar{t})^j = B_{0,j}(\bar{\omega}\cdot\bar{t}; -)$, we use Corollary 2.3 to integrate with respect to $t_1, \ldots, t_N$ and obtain

$$\int_{\mathbb{Z}_p^N} \frac{(x + \bar{\omega}\cdot\bar{t})^N}{\langle x + \bar{\omega}\cdot\bar{t}\rangle^s}\, d\bar{t} = x^N \langle x \rangle^{-s} \sum_{j=0}^{\infty} \binom{N-s}{j} \omega_1 \cdots \omega_N B_{N,j}(0; \bar{\omega}) x^{-j}. \tag{4.2}$$

Dividing both sides by $\omega_1 \cdots \omega_N (s-1)\cdots(s-N)$ gives the result. □

Theorem 4.2. *Suppose $x, \omega_1, \ldots, \omega_N \in \mathbb{C}_p^\times$ and $|x|_p > ||\bar{\omega}||_p$. Then*

$$\begin{aligned} G_{p,N}(x;\bar{\omega}) &= \frac{(-1)^{N+1}}{N!} B_{N,N}(x;\bar{\omega}) \log_p x \\ &+ \sum_{j=0}^{N-1} \frac{(-1)^N}{j!(N-j)!} \left(1 + \frac{1}{2} + \cdots + \frac{1}{N-j}\right) B_{N,j}(0;\bar{\omega}) x^{N-j} \\ &+ \sum_{j=N+1}^{\infty} \frac{(-1)^j (j-N-1)!}{j!} B_{N,j}(0;\bar{\omega}) x^{N-j}. \end{aligned}$$

Equivalently, by Eq. (3.10),

$$\begin{aligned} G_{p,N}(x;\bar{\omega}) &= -\zeta_{p,N}(0,x;\bar{\omega}) \log_p x \\ &+ \sum_{j=0}^{N-1} \frac{(-1)^N}{j!(N-j)!} \left(1 + \frac{1}{2} + \cdots + \frac{1}{N-j}\right) B_{N,j}(0;\bar{\omega}) x^{N-j} \\ &+ \sum_{j=N+1}^{\infty} \frac{(-1)^j (j-N-1)!}{j!} B_{N,j}(0;\bar{\omega}) x^{N-j}. \end{aligned}$$

Proof. Rewrite the expansion of Theorem 4.1 as

$$\begin{aligned} \zeta_{p,N}(s,x;\bar{\omega}) &= \frac{x^N \langle x \rangle^{-s}}{(s-1)\cdots(s-N)} \sum_{j=0}^{\infty} \binom{N-s}{j} B_{N,j}(0;\bar{\omega}) x^{-j}. \\ &= \langle x \rangle^{-s} \left[\sum_{j=0}^{N} \frac{(-1)^j}{j!(s-1)\cdots(s-N+j)} B_{N,j}(0;\bar{\omega}) x^{N-j} \right. \\ &+ \left. \sum_{j=N+1}^{\infty} \frac{(-1)^N (j-N)!}{j!} \binom{-s}{j-N} B_{N,j}(0;\bar{\omega}) x^{N-j} \right] \end{aligned} \tag{4.3}$$

and evaluate the partial derivative with respect to s at $s = 0$. □

As mentioned in the Introduction, this p-adic Laurent series expansion for $G_{p,N}(x;\bar{\omega})$ agrees exactly with the asymptotic expansion (1.4) for $\log(\Gamma_N(x;\bar{\omega}))$ given by Barnes with the fortunate exception that the infinite series converges p-adically for large x and the error term $R_{N,M}$ vanishes. This phenomenon was first observed in the case $N = 1$ by Diamond ([D], Theorem 6).

5. Parameters in $\mathbb{Q}_2$

We have defined our p-adic multiple zeta functions $\zeta_{p,N}(s,x;\bar{\omega})$ as an integral involving the function $\langle z \rangle$, which is the projection onto the third factor in the direct product decomposition (2.16). This formulation allows us to define our zeta function in the greatest generality. However, we remark that in the special case when $p = 2$ and the parameters $x, \omega_1, \ldots, \omega_N$ lie in $\mathbb{Q}_2$, there are advantages to replacing the decomposition (2.16) with the decomposition

$$\mathbb{Q}_2^\times \simeq 2^{\mathbb{Z}} \times \{1, -1\} \times D_4, \tag{5.1}$$

where $D_4 = \{x \in \mathbb{Q}_2 : |x - 1|_2 < 1/2\}$, given by

$$x = 2^{\nu_2(x)} \cdot \hat{(x)}_4 \cdot \langle x \rangle_4 \mapsto (p^{\nu_2(x)}, \hat{(x)}_4, \langle x \rangle_4). \tag{5.2}$$

One advantage is that this definition of $\langle z \rangle_4$ then agrees with Washington's [W] use of the notation $\langle z \rangle$ when $p = 2$ and $z \in \mathbb{Q}_2$. The other advantage is that $\langle -z \rangle_4 = \langle z \rangle_4$ (whereas $\langle -z \rangle = -\langle z \rangle$ when $p = 2$), so that if we replace $\langle z \rangle$ with $\langle z \rangle_4$ in our definition of $\zeta_{2,N}$ the resulting function would satisfy the reflection functional equation (Theorem 3.2(iii)) for all s for which zeta is defined, rather than just on a small disc about $s = 0$ (see Eq. (3.7) and the ensuing remarks). Since $\langle z \rangle_4^s = \langle z \rangle^s$ on a disc about $s = 0$, replacing $\langle z \rangle$ with $\langle z \rangle_4$ in the definition of $\zeta_{2,N}$ would not change the definition of $G_{2,N}$.

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