On the behavior of some two-variable p-adic L-functions

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We give some p-adic integral representations for the two-variable p-adic L-functions introduced recently by G. Fox. For powers of the Teichmüller character, we use the integral representation to extend the L-function to a larger domain, in which it is a meromorphic function in the first variable and an analytic element in the second. These integral representations imply systems of congruences for the generalized Bernoulli polynomials, improving previous results of Fox, Gunaratne, and the author; they also lead to generalizations of some formulas of Diamond and of Ferrero and Greenberg for p-adic L-functions in terms of the p-adic gamma and log gamma functions.

Keywords. p-adic L-functions; generalized Bernoulli polynomials; Kummer congruences.

1. INTRODUCTION.

In a recent article [7] G. Fox defined a two-variable p-adic L-function $L_p(s,t,\chi)$ for Dirichlet characters χ, with the property that

$$L_p(1-m,t,\chi) = -\frac{B_{m,\chi\omega^{-m}}(qt) - \chi\omega^{-m}(p)p^{m-1}B_{m,\chi\omega^{-m}}(p^{-1}qt)}{m} \tag{1.1}$$

for positive integers m and $t \in \mathbb{C}_p$ with $|t| \le 1$, where $q = p$ if $p > 2$ and $q = 4$ if $p = 2$, and $B_{m,\chi\omega^{-m}}(x)$ is a generalized Bernoulli polynomial (see §2 for definitions). He proved that $L_p(s,t,\chi)$ is analytic in s and t for $s \in \mathbb{C}_p$ with $|s| < qp^{-1/(p-1)}$ (except for $s = 1$ if $\chi = 1$) and $t \in \mathbb{C}_p$ with $|t| \le 1$. Therefore $L_p(s,t,\chi)$ is a natural extension of the Kubota-Leopoldt p-adic L-function $L_p(s,\chi)$ ([10], [9]), which is the function obtained by putting $t = 0$ in $L_p(s,t,\chi)$. Fox developed some analytic and arithmetic properties of $L_p(s,t,\chi)$ and used these properties to prove strong general systems of congruences for values of generalized Bernoulli polynomials, which generalize the classical Kummer congruences [1], [2], and congruences of H. S. Gunaratne [8].

Using a different approach, we proved similar congruences [13] for values of Bernoulli polynomials at arguments corresponding to t values outside the disc of analyticity of $L_p(s,t,\chi)$. In an effort to explain this phenomenon, we now show that when $\chi = \omega^k$ is a power of the Teichmüller character, $L_p(s,t,\chi)$ may be extended to a domain which includes $t \in q^{-1}\mathbb{Z}_p$, so that is an analytic function in s (except at $s = 1$ when $\chi = 1$), and an analytic element in t on $q^{-1}\mathbb{Z}_p$. This extension interpolates Bernoulli polynomial values from (1.1) and from the congruences in [13]. We deduce this result from a p-adic integral representation for $L_p(s, t, \chi)$, which we give in section 3 for Dirichlet characters of p-power conductor. In section 4 we give a slightly different integral representation in the case of characters whose conductor is not a power of p. These integral representations then produce systems of congruences which generalize the results of [7], [8], and [13].

Whereas the usual integral representations for $L_p(s, \chi)$ involve "fixed" p-adic measures (the regularized Bernoulli measures) on a space X^* which varies with the character χ, we employ an approach for $L_p(s, t, \chi)$ in which the space of integration $\mathbb{Z}_p^\times$ is fixed and the measure varies with the character, as well as with t. This approach allows us to eliminate the regularizing factor $(1 - \chi(b)\langle b\rangle^{1-s})$ from our formulas for $L_p(s, t, \chi)$ in the case of characters χ whose conductor is not a power of p. The elimination of this factor facilitates the study of the analytic properties of $L_p(s, t, \chi)$ for such characters χ. As an illustration we give a generalization to $L_p(s, t, \chi)$ of a formula of J. Diamond [4] and of B. Ferrero and R. Greenberg [6] for $L_p'(0, \chi)$. Of course the regularizing factor cannot be removed in the case of powers of the Teichmüller character, because its vanishing at $s = 1$ is needed to "correct" the singularity of L_p at $s = 1$ when χ is the trivial character; indeed this factor vanishes for all b only when $\chi = 1$ and $s = 1$.

There is some flexibility in these integral formulas with respect to the characters involved, in that both the integrand and the measure are defined in terms of Dirichlet characters. For example, in §3 we show that for powers of the Teichmüller character ω, we can define $L_p(s, t, \omega^k)$ (up to the regularizing factor) as a p-adic Γ-transform of the measure $\omega^{i-1}\mu_{\omega^j,b,x}$ for any choice of integers i, j such that $i + j = k$. That these $\phi(q)$ different definitions all define the same function for $t \in \mathbb{Z}_p$ can be verified by straightforward computations with the associated formal power series; or, since Fox has shown that $L_p(s, t, \chi)$ is the unique function analytic in s satisfying (1.1), by observing that these integral representations are analytic in s and by definition agree with $L_p(s, t, \chi)$ on sets with limit points in $\mathbb{Z}_p$. In the same way, when $p = 2$ and χ is not of the second kind, we show in §5 that there are two natural ways to extend the function $L_2(s, t, \chi)$ to values of t in $\frac{1}{2}\mathbb{Z}_2$ which satisfy (1.1) when $t \in \mathbb{Z}_2$, but for $t \in (\frac{1}{2}\mathbb{Z}_2) \setminus \mathbb{Z}_2$ these two extensions are not equal.

2. Preliminaries.

Throughout this paper, p will denote a prime number, $\mathbb{Z}_p$ the ring of p-adic integers, $\mathbb{Z}_p^\times$ the multiplicative group of units in $\mathbb{Z}_p$, $\mathbb{Q}_p$ the field of p-adic numbers, $\mathbb{C}_p$ the completion of an algebraic closure of $\mathbb{Q}_p$, and $\mathfrak{O}$ the subring $\mathfrak{O} = \{x \in \mathbb{C}_p : |x| \leq 1\}$ of $\mathbb{C}_p$, where $|\cdot|$ denotes the absolute value on $\mathbb{C}_p$ normalized so that $|p| = p^{-1}$. We let ord_p denote the additive valuation on $\mathbb{C}_p$ normalized so $\mathrm{ord}_p p = 1$, so $|x| = p^{-\mathrm{ord}_p x}$ for all $x \neq 0$. The integer q is defined by $q = p$ if $p > 2$ and $q = 4$ if $p = 2$. The Teichmüller character ω on $\mathbb{Z}_p^\times$ is defined by setting $\omega(a)$ to be the unique $\phi(q)$-th root of unity congruent to a modulo $q\mathbb{Z}_p$, and we define $\langle a \rangle$ by $a = \omega(a) \cdot \langle a \rangle$ for $a \in \mathbb{Z}_p^\times$. Following [7], if $a \in \mathbb{Z}_p^\times$ and $t \in \mathfrak{O}$, we extend these definitions by $\omega(a + qt) = \omega(a)$ and $a + qt = \omega(a) \cdot \langle a + qt \rangle$; we also consider ω as a Dirichlet character of conductor q by setting $\omega(n) = 0$ if p divides n. If K is a finite extension of $\mathbb{Q}_p$ then $\mathfrak{O}_K = \mathfrak{O} \cap K$ will denote its ring of integers, and $\mathfrak{O}_K[T-1]$ and $\mathfrak{O}_K[[T-1]]$ denote respectively the ring of polynomials and of formal power series in the indeterminate $(T-1)$ over $\mathfrak{O}_K$.

The set $\Lambda_{\mathfrak{O}_K}$ of all $\mathfrak{O}_K$-valued measures on $\mathbb{Z}_p$ forms a ring under addition and convolution of measures. This ring is isomorphic to the formal power series ring $\mathfrak{O}_K[[T-1]]$ by means of the isomorphism $\Lambda_{\mathfrak{O}_K} \to \mathfrak{O}_K[[T-1]]$ defined by $\mu \mapsto h$, where

$$h(T) = \int_{\mathbb{Z}_p} T^a \, d\mu(a). \tag{2.1}$$

The linear operator φ defined by

$$\varphi h(T) = h(T) - \frac{1}{p} \sum_{\zeta^p = 1} h(\zeta T) \tag{2.2}$$

is well-defined and stable on rational functions, and also on $\mathfrak{O}_K[[T-1]]$ (cf. [12]). If $h(e^t) = \sum a_n t^n/n!$, write $(\varphi h)(e^t) = \sum \hat{a}_n t^n/n!$. Then if μ is the measure on $\mathfrak{O}_K$ corresponding to $h \in \mathfrak{O}_K[[T-1]]$, then for any nonnegative integer m we have $a_m, \hat{a}_m \in \mathfrak{O}_K$ given by

$$a_m = \int_{\mathbb{Z}_p} a^m \, d\mu(a) \qquad \text{and} \qquad \hat{a}_m = \int_{\mathbb{Z}_p^\times} a^m \, d\mu(a) \tag{2.3}$$

(cf [12]). Furthermore we have $\hat{a}_m = a_m - p^m a_m^*$ where the $a_m^* \in \mathfrak{O}_K$ are defined by $(\psi h)(e^t) = \sum a_n^* t^n/n!$, where Dwork's ψ operator [5] is defined on $\mathfrak{O}_K[[T-1]]$ by

$$(\psi h)(T) = \frac{1}{p} \sum_{Z^p = T} h(Z). \tag{2.4}$$

The Bernoulli polynomials $B_n(x)$ are defined by the generating function

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, \tag{2.5}$$

and for any primitive Dirichlet character χ of conductor f the generalized Bernoulli polynomials $B_{n,\chi}(x)$ are defined by

$$\left(\sum_{a=1}^{f} \frac{\chi(a) t e^{at}}{e^{ft} - 1} \right) e^{xt} = \sum_{n=0}^{\infty} B_{n,\chi}(x) \frac{t^n}{n!}. \tag{2.6}$$

When the Dirichlet L-function defined by $L(s,\chi) = \sum_{n=1}^{\infty} \chi(n) n^{-s}$ for $\Re(s) > 1$ is analytically continued to $\mathbb{C}$, we have

$$L(1-m, \chi) = -B_{m,\chi}(0)/m \tag{2.7}$$

for all nonnegative integers m (cf. [9]). Its p-adic analogue is the Kubota-Leopoldt p-adic L-function $L_p(s,\chi)$ ([9], [10]), which is the unique analytic function on $\mathfrak{D} = \{s \in \mathbb{C}_p : |s| \leq qp^{-1/(p-1)}\}$ (except for a simple pole at $s = 1$ when $\chi = 1$) for which

$$L_p(1-m, \chi) = -(1 - \chi_m(p) p^{m-1}) B_{m,\chi_m}(0)/m \tag{2.8}$$

for positive integers m, where the symbol χ_m denotes the Dirichlet character $\chi\omega^{-m}$. The two-variable p-adic L-function $L_p(s,t,\chi)$ defined by Fox in [7] generalizes this, being analytic for $t \in \mathfrak{O}$ and $s \in \mathfrak{D}$, except at $s = 1$ if $\chi = 1$, and its values at nonpositive integer values of s are given by (1.1), which reduces to (2.8) when $t = 0$. In its construction, based on the method of [9], a function $A_\chi(s,t)$, analytic for $s \in \mathfrak{D}$ and $t \in \mathfrak{O}$, is defined so as to interpolate values of generalized Bernoulli polynomials at nonnegative integer values of s, and then $L_p(s,t,\chi)$ is defined by

$$L_p(s,t,\chi) = \frac{1}{s-1} A_\chi(1-s, t). \tag{2.9}$$

Our congruences are expressed in terms of the difference operator Δ_c where c is a nonnegative integer, which operates on sequences $\{a_m\}$ by

$$\Delta_c\, a_m = a_{m+c} - a_m. \tag{2.10}$$

The powers Δ_c^k of Δ_c are defined by $\Delta_c^0 =$ identity and $\Delta_c^k = \Delta_c \circ \Delta_c^{k-1}$ for positive integers k, so that

$$\Delta_c^k\, a_m = \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} a_{m+jc} \tag{2.11}$$

for all nonnegative integers k. To define binomial coefficient operators $\binom{D}{k}$ associated to an operator D (cf. [8]), we write the binomial coefficient

$$\binom{X}{k} = \frac{X(X-1)\cdots(X-k+1)}{k!} \tag{2.12}$$

for $k \geq 0$ as a polynomial in X, and replace X by D.

3. Characters of p-power Conductor.

Using a modification of a regularized Bernoulli measure on $\mathbb{Z}_p^\times$, in this section we give an integral representation for $L_p(s,t,\chi)$ in the case where the conductor of χ is a power p. We recall that Dwork's shift map $x \mapsto x'$ is defined [5] for $x \in \mathbb{Z}_p$ by the relation $px' - x = \mu_x \in \{0, 1, ..., p-1\}$, so that μ_x is the least nonnegative integer representative of the class of $-x$ modulo $p\mathbb{Z}_p$.

Theorem 3.1. *Suppose that χ is a primitive Dirichlet character whose conductor f is a power of p. If $x \in \mathbb{Z}_p$ and b is any positive integer such that $(b,p) = 1$ and $(b-1)x \equiv 0 \pmod{f\mathbb{Z}_p}$, then the power series*

$$h_{\chi,b,x}(T) = b\chi(b)T^{-bx}\left(\sum_{a=1}^{f} \frac{\chi(a)T^{ab}}{T^{bf}-1}\right) - T^{-x}\sum_{a=1}^{f} \frac{\chi(a)T^{a}}{T^{f}-1}$$

lies in $\mathfrak{O}_K[[T-1]]$, where $K = \mathbb{Q}_p(\chi)$, and therefore the associated measure $\mu = \mu_{\chi,b,x}$ is an $\mathfrak{O}_K$-valued measure on $\mathbb{Z}_p$. Then for $i \in \mathbb{Z}$, $t \in \mathbb{Z}_p$ and $s \in \mathfrak{D}$ we have

$$\int_{\mathbb{Z}_p^\times} \omega^{i-1}(a)\langle a\rangle^s\, d\mu_{\chi,b,x}(a) = (1 - \chi\omega^i(b)\langle b\rangle^{s+1})L_p(-s,t,\chi\omega^i)$$

for any such b, with $x = -qt$.

REMARKS. The measure $-\mu_{1,b,1}$ defined above is a regularized Bernoulli measure on $\mathbb{Z}_p^\times$. For any $x \in \mathbb{Z}_p$, if $b \neq 1$ is chosen so that $(b-1)x \equiv 0 \pmod{f\mathbb{Z}_p}$, the measure $\mu_{\chi,b,x}$ is an $\mathfrak{O}_K$-valued measure, so by writing $x = -qt$ we can take this integral as defining for $t \in q^{-1}\mathbb{Z}_p$ a meromorphic function of s, analytic for $s \neq 1$, which agrees with the function $L_p(s,t,\chi\omega^i)$ defined by Fox if $t \in \mathbb{Z}_p$. When b can be chosen so that we also have $\chi\omega^i(b) \neq 1$, then this extended function will be analytic for all $s \in \mathfrak{D}$; this is indeed the case for all $x \in \mathbb{Z}_p$ when $\chi = 1$ and $\omega^i \neq 1$, or when $p = 2$, $\chi = \omega$, $\omega^i = 1$, and $x \in 2\mathbb{Z}_2$.

Proof. We assume $(b,p) = 1$ and write $h(T) = h_{\chi,b,x}(T) = g(T)/(T^{bf} - 1)$, where

$$g(T) = b\chi(b)T^{-bx}\left(\sum_{a=1}^{f}\chi(a)T^{ab}\right) - T^{-x}\sum_{a=1}^{bf}\chi(a)T^a \in \mathfrak{O}_K[[T-1]]. \tag{3.1}$$

If $\zeta^f = 1$, then

$$\begin{aligned} g(\zeta) &= b\chi(b)\zeta^{-bx}\left(\sum_{a=1}^{f}\chi(a)\zeta^{ab}\right) - \zeta^{-x}\sum_{a=1}^{bf}\chi(a)\zeta^a \\ &= b\left(\zeta^{-bx}\sum_{a=1}^{f}\chi(ab)\zeta^{ab} - \zeta^{-x}\sum_{a=1}^{f}\chi(a)\zeta^a\right) \\ &= b\zeta^{-x}(\zeta^{(1-b)x} - 1)\left(\sum_{a=1}^{f}\chi(a)\zeta^a\right) = 0, \end{aligned} \tag{3.2}$$

provided that $(b-1)x \equiv 0 \pmod{f\mathbb{Z}_p}$. For such integers b, this implies that $T^f - 1$ divides $g(T)$ in $\mathfrak{O}_K[[T-1]]$. Since $T^{bf} - 1$ is also divisible by $T^f - 1$ in $\mathfrak{O}_K[T-1]$ and the quotient is a unit in $\mathfrak{O}_K[[T-1]]^\times$, we see that $h_{\chi,b,x} \in \mathfrak{O}_K[[T-1]]$. Therefore when $(b-1)x \equiv 0 \pmod{f\mathbb{Z}_p}$, the measure $\mu = \mu_{\chi,b,x}$ is an $\mathfrak{O}_K$-valued measure, and by definition (2.1),

$$h(T) = \int_{\mathbb{Z}_p} T^a \, d\mu(a). \tag{3.3}$$

Substituting $T = e^u$ and comparing coefficients of $u^m/m!$ yields

$$a_m = (b^{m+1}\chi(b) - 1)\frac{B_{m+1,\chi}(-x)}{m+1} = \int_{\mathbb{Z}_p} a^m \, d\mu(a). \tag{3.4}$$

If χ is nontrivial then $\chi(a) = 0$ whenever p divides a, and therefore $\psi h_{\chi,b,x} = 0$ when $x \in p\mathbb{Z}_p$ (cf. [13], eq. (4.8)). If χ is the trivial character then

$$h_{\chi,b,x}(T) = \frac{bT^{b(1-x)}}{T^b - 1} - \frac{T^{(1-x)}}{T-1}, \tag{3.5}$$

and therefore by ([13], Lemma 3.1) we have

$$\psi h_{\chi,b,x}(T)=\frac{bT^{b(1-x)'}}{T^b-1}-\frac{T^{(1-x)'}}{T-1} \tag{3.6}$$

for any $x\in\mathbb{Z}_p$. Using the identity $(1-x)'=1-x'$ for any $x\in\mathbb{Z}_p$, we have $\psi h_{\chi,b,x}=\chi(p)h_{\chi,b,x'}$ if either $\chi=1$ and $x\in\mathbb{Z}_p$, or if $\chi\neq 1$ and $x\in p\mathbb{Z}_p$. This means that

$$a_m-p^m a_m^*=(b^{m+1}\chi(b)-1)\frac{B_{m+1,\chi}(-x)-\chi(p)p^m B_{m+1,\chi}(-x')}{m+1}=\int_{\mathbb{Z}_p^\times}a^m\,d\mu(a) \tag{3.7}$$

if $\chi=1$ and $x\in\mathbb{Z}_p$, or if $\chi\neq 1$ and $x\in p\mathbb{Z}_p$, when $(b-1)x\in f\mathbb{Z}_p$.

Now write $x=-qt$ with $t\in\mathbb{Z}_p$, so that $x'=-p^{-1}qt$. Then by comparing (3.7) with (1.1) we have

$$\int_{\mathbb{Z}_p^\times}a^m\,d\mu(a)=(1-\chi(b)b^{m+1})L_p(-m,t,\chi\omega^{m+1}). \tag{3.8}$$

This says that if s is a nonnegative integer congruent to $i-1$ modulo $\phi(q)$, then

$$\int_{\mathbb{Z}_p^\times}\omega^{i-1}(a)\langle a\rangle^s\,d\mu(a)=(1-\chi\omega^i(b)\langle b\rangle^{s+1})L_p(-s,t,\chi\omega^i). \tag{3.9}$$

As the p-adic Γ-transform of the measure $\omega^{i-1}\mu$, the integral on the left is analytic for $s\in\mathfrak{D}$ (cf. [11], Corollary 12.5). If $t\in\mathbb{Z}_p$ and $\chi\omega^i\neq 1$, then $L_p(s,t,\chi\omega^i)$ is analytic for $s\in\mathfrak{D}$ by Theorem 3.13 in [7], so the right side of (3.9) is analytic in s, being the product of two analytic functions. And if $\chi\omega^i=1$ then for all $t\in\mathbb{Z}_p$,

$$(1-\chi\omega^i(b)\langle b\rangle^{s+1})L_p(-s,t,\chi\omega^i)=\left(\frac{\langle b\rangle^{s+1}-1}{s+1}\right)A_{\chi\omega^i}(s+1,t) \tag{3.10}$$

is analytic for $s\in\mathfrak{D}$, being the product of two analytic functions (cf. (2.9) above and Theorem 3.13 of [7]). Thus in any case both sides of (3.9) are analytic for $s\in\mathfrak{D}$; since they agree on the set $\{s\in\mathbb{Z}: s\geq 0, s\equiv i-1 \pmod{\phi(q)}\}$, their equality holds for all $s\in\mathfrak{D}$.

This integral representation leads directly to the following congruences for the generalized Bernoulli polynomials.

Corollary 3.2. *Suppose that χ is a nontrivial Dirichlet character whose conductor f is a power of p. If c, k, m are any positive integers, then for all $t\in\mathbb{Z}_p\cap pq^{-2}f\mathbb{Z}_p$,*

$$\Delta_c^k\left\{\frac{B_{m,\chi_m}(qt)-\chi_m(p)p^{m-1}B_{m,\chi_m}(p^{-1}qt)}{m}\right\}\equiv 0 \pmod{\frac{1}{2}c^k q^k\mathfrak{D}_K}$$

and

$$\binom{\rho\Delta_c}{k}\left\{\frac{B_{m,\chi_m}(qt)-\chi_m(p)p^{m-1}B_{m,\chi_m}(p^{-1}qt)}{m}\right\}\in\frac{1}{2}\mathfrak{O}_K$$

for any $\rho\in(cq)^{-1}\mathbb{Z}_p$.

Proof. By (1.1) we have $L_p(1-m,t,\chi)=\beta_{m,\chi}(t)$ for $t\in\mathfrak{O}$, where (as in [7])

$$\beta_{m,\chi}(t)=-\frac{B_{m,\chi_m}(qt)-\chi_m(p)p^{m-1}B_{m,\chi_m}(p^{-1}qt)}{m};\tag{3.11}$$

the stated congruences then involve values of $\beta_{m,\chi}(t)$ for values of $t\in\mathbb{Z}_p$. Let $x=-qt$ with $t\in pq^{-2}f\mathbb{Z}_p$ and observe that $-x'=p^{-1}qt$. Since we assume $t\in pq^{-2}f\mathbb{Z}_p$ we have $(b-1)qt\in f\mathbb{Z}_p$ for any positive integer b with $(b,p)=1$, so we choose b so that $\chi(b)\not\equiv 1 \pmod{q\mathbb{Z}_p}$ and $\langle b\rangle\equiv 1 \pmod{c^kq^k\mathbb{Z}_p}$. Then writing

$$1-\chi(b)\langle b\rangle^m=(1-\chi(b))+\chi(b)(1-\langle b\rangle^m)\tag{3.12}$$

reveals that $\mathrm{ord}_p(1-\chi(b)\langle b\rangle^{m+1})=\mathrm{ord}_p(1-\chi(b))=\mathrm{ord}_p 2$, and that $(1-\chi(b)\langle b\rangle^m) \pmod{c^kq^k\mathbb{Z}_p}$ is independent of m. So if $\mu=\mu_{\chi,b,x}$ is the $\mathfrak{O}_K$-valued measure defined in Theorem 3.1, we get

$$\begin{aligned}(1-\chi(b)\langle b\rangle^m)\Delta_c^k\beta_{m,\chi}(t)&\equiv\Delta_c^k\{(1-\chi(b)\langle b\rangle^m)\beta_{m,\chi}(t)\}\\&=\Delta_c^k\{(1-\chi(b)\langle b\rangle^m)L_p(1-m,t,\chi)\}\\&=\Delta_c^k\int_{\mathbb{Z}_p^\times}\omega^{-1}(a)\langle a\rangle^{m-1}\,d\mu(a)\\&=\int_{\mathbb{Z}_p^\times}\omega^{-1}(a)\langle a\rangle^{m-1}(\langle a\rangle^c-1)^k\,d\mu(a)\\&\equiv 0\qquad(\mathrm{mod}\ c^kq^k\mathfrak{O}_K),\end{aligned}\tag{3.13}$$

since $\langle a\rangle^c\equiv 1 \pmod{cq\mathbb{Z}_p}$. Observing that $\mathrm{ord}_p(1-\chi(b)\langle b\rangle^m)=\mathrm{ord}_p 2$ completes the proof of the first part. The same reasoning shows that

$$(1-\chi(b)\langle b\rangle^m)\binom{\rho\Delta_c}{k}\beta_{m,\chi}(t)\equiv\int_{\mathbb{Z}_p^\times}\omega^{-1}(a)\langle a\rangle^{m-1}\binom{\rho(\langle a\rangle^c-1)}{k}\,d\mu(a)\equiv 0\quad(\mathrm{mod}\ \mathfrak{O}_K),\tag{3.14}$$

since $\binom{x}{k}\in\mathbb{Z}_p$ when $x\in\mathbb{Z}_p$ and $\rho(\langle a\rangle^c-1)\in\mathbb{Z}_p$ when $\rho\in(cq)^{-1}\mathbb{Z}_p$, from which the second statement follows.

For the characters involved, this corollary gives a modest improvement of Theorem 4.10 and Theorem 4.12 of [7] by relaxing its restriction on $|t|$ by a factor of q; for powers of ω it improves

Theorem 4.2 of [13] by removing the restriction that $\phi(q)$ divides c. Putting $t=0$ and $\rho=p^{-1}$ for powers of ω reduces this result to [8], Theorem 3.1.

In ([13], Theorem 3.2) we used (3.7) above with $\chi=1$ to prove that for any $x \in \mathbb{Z}_p$,

$$\Delta_c^k \left\{ \frac{B_m(x) - p^{m-1}B_m(x')}{m} \right\} \equiv 0 \pmod{\frac{1}{2}c^k q^k \mathbb{Z}_p}, \tag{3.15}$$

and

$$\binom{p^{-r}\Delta_c}{k} \left\{ \frac{B_m(x) - p^{m-1}B_m(x')}{m} \right\} \in \frac{1}{2}\mathbb{Z}_p \tag{3.16}$$

when $p^{-r} \in (cq)^{-1}\mathbb{Z}_p$, under the additional hypotheses that $\phi(q)$ divides c but doesn't divide m. Using the identity $B_{n,1}(x) = (-1)^n B_n(-x)$ and (3.11), we observe that

$$\beta_{m,\omega^m}(t) = (-1)^{m+1} \frac{B_m(x) - p^{m-1}B_m(x')}{m}, \tag{3.17}$$

where $x = -qt$ with $t \in \mathbb{Z}_p$. We would therefore like to be able to interpret the congruences (3.15), (3.16) as congruences for values of $L_p(1-m, t, \omega^m)$.

In Theorem 3.1 we have observed that if $\chi = 1$, the power series $h_{\chi,b,x}$ lies in $\mathbb{Z}_p[[T-1]]$ for any $x \in \mathbb{Z}_p$ and any positive integer b with $(b,p)=1$; this means that $\mu_{1,b,x}$ is a $\mathbb{Z}_p$-valued measure on $\mathbb{Z}_p$, and (3.7) is in fact valid for all $x \in \mathbb{Z}_p$ when $\chi = 1$. Therefore, if $x = -qt$ with $t \in q^{-1}\mathbb{Z}_p$, we can take the integral

$$(1 - \omega^i(b)\langle b\rangle^{1-s})^{-1} \int_{\mathbb{Z}_p^\times} \omega^{i-1}(a)\langle a\rangle^{-s}\, d\mu_{1,b,x}(a) = L_p(b, s, t, \omega^i) \tag{3.18}$$

as defining a function $L_p(b,s,t,\omega^i)$ which is equal to the function $L_p(s,t,\omega^i)$ defined by Fox for $t \in \mathbb{Z}_p$. We claim that for $t \in q^{-1}\mathbb{Z}_p$ this definition is independent of b. To show this, we observe that the integral in (3.18) is analytic for $s \in \mathfrak{D}$, being the p-adic Γ-transform of $\omega^{i-1}\mu_{1,b,x}$. Therefore $L_p(b,s,t,\omega^i)$ is analytic for $s \in \mathfrak{D}$ if $\omega^i \neq 1$, and $(s-1)L_p(b,s,t,\omega^i)$ is analytic for $s \in \mathfrak{D}$ if $\omega^i = 1$. By (3.7),

$$L_p(b, -s, t, \omega^i) = -\frac{B_{s+1,1}(-x) - p^s B_{s+1,1}(-x')}{s+1} \tag{3.19}$$

whenever s is a nonnegative integer congruent to $i-1$ modulo $\phi(q)$. So if $(b_1,p)=(b_2,p)=1$, the functions $(s-1)L_p(b_1,s,t,\omega^i)$ and $(s-1)L_p(b_2,s,t,\omega^i)$ are both analytic for $s \in \mathfrak{D}$ and agree on a set of s values having a limit point in $\mathfrak{D}$, so they must be equal. Therefore we may omit the parameter b and write $L_p(s,t,\omega^i)$ for the function defined for all $t \in q^{-1}\mathbb{Z}_p$ by (3.18).

To give an indication that the extension defined by (3.18) is somewhat natural, we show that for any $s \in \mathfrak{D}$, $L_p(s,t,\omega^i)$ as a function of t is in fact an analytic element of support $q^{-1}\mathbb{Z}_p$, that is, a uniform limit on that set of a sequence of rational functions which have no poles on that set.

THEOREM 3.3. *Suppose that $\chi = \omega^i$ is a power of the Teichmüller character. Then the function $L_p(s,t,\omega^i)$ defined by*

$$L_p(s,t,\omega^i) = (1-\omega^i(b)\langle b\rangle^{1-s})^{-1}\int_{\mathbb{Z}_p^\times} \omega^{i-1}(a)\langle a\rangle^{-s}\, d\mu_{1,b,x}(a),$$

where $x=-qt$ and b is a positive integer such that $(b,p)=1$ and $b\neq 1$, is an analytic function of s on $\mathfrak{D}$ (except at $s=1$ if $\omega^i=1$) for each $t\in q^{-1}\mathbb{Z}_p$, and is an analytic element in t on $q^{-1}\mathbb{Z}_p$ for all $s\in\mathfrak{D}$ (except for $s=1$ if $\omega^i=1$). Furthermore, for $t\in\mathbb{Z}_p$ we have

$$L_p(1-m,t,\omega^i) = -\frac{B_{m,\omega^{i-m}}(qt) - \omega^{i-m}(p)p^{m-1}B_{m,\omega^{i-m}}(p^{-1}qt)}{m}$$

for all positive integers m, and for $t\in q^{-1}\mathbb{Z}_p$ we have

$$L_p(1-m,t,\omega^i) = (-1)^{m+1}\frac{B_m(x) - p^{m-1}B_m(x')}{m}$$

for all positive integers $m \equiv i \pmod{\phi(q)}$.

Proof. Aside from the claim that $L_p(s,t,\omega^i)$ is an analytic element in t, all the assertions of the Theorem are immediate from the preceding comments. Let's define the polynomial $g(x) = (x(x-1)(x-2)\cdots(x-(p-1)))+x$. It is easily observed that $g(x)\equiv x^p \pmod{p\mathbb{Z}[x]}$, and $g(x)=x$ if and only if $x\in\{0,1,2,...,p-1\}$. Now for any $x\in\mathbb{Z}_p$, we have $-x\equiv\mu_x \pmod{p\mathbb{Z}_p}$, and if $-x\equiv\mu_x \pmod{p^r\mathbb{Z}_p}$ then $g(-x)\equiv\mu_x \pmod{p^{r+1}\mathbb{Z}_p}$, because $g'(x)\in p\mathbb{Z}_p[x]$. Therefore, the sequence of polynomials $g(-x), g(-g(-x)), g(-g(-g(-x)))$,... converges uniformly to the function $x\mapsto\mu_x$ on $\mathbb{Z}_p$; since $x'=(x+\mu_x)/p$, the function $x\mapsto x'$ is an analytic element on $\mathbb{Z}_p$. For any positive integer m, by (2.3) we have

$$\int_{\mathbb{Z}_p^\times} a^m\, d\mu_{1,b,x}(a) = a_m - p^m a_m^* \tag{3.20}$$

where $h(e^u) = \sum a_n u^n/n!$ and $(\psi h)(e^u) = \sum a_n^* u^n/n!$, and h, ψh are given by (3.5), (3.6). We see that a_m is a polynomial in x and a_m^* is a polynomial in x', so the integral in (3.20) represents an analytic element in x on $\mathbb{Z}_p$ for any nonnegative integer m.

Now let $s \in \mathfrak{D}$. Since $\langle a \rangle \equiv 1 \pmod{q\mathbb{Z}_p}$ for all $a \in \mathbb{Z}_p^\times$, the series $\langle a \rangle^s = \sum_{n=0}^{\infty} \binom{s}{n} (\langle a \rangle - 1)^n$ converges uniformly for $a \in \mathbb{Z}_p^\times$, and therefore the sum

$$\int_{\mathbb{Z}_p^\times} \omega^{i-1}(a) \langle a \rangle^s \, d\mu_{1,b,x}(a) = \sum_{n=0}^{\infty} \binom{s}{n} \int_{\mathbb{Z}_p^\times} \omega^{i-1}(a)(\langle a \rangle - 1)^n \, d\mu_{1,b,x}(a) \tag{3.21}$$

converges uniformly in x on $\mathbb{Z}_p$, since $\mu_{1,b,x}$ is a $\mathbb{Z}_p$-valued measure for all $x \in \mathbb{Z}_p$. For odd primes p, we have $\omega(a) = \lim_{r\to\infty} a^{p^r}$ uniformly on $\mathbb{Z}_p$, and thus $\langle a \rangle = \lim_{r\to\infty} a^{(p-2)p^r+1}$ uniformly on $\mathbb{Z}_p^\times$. Therefore, each integrand in the sum of integrals on the right side of (3.21) is a uniform limit on $\mathbb{Z}_p^\times$ of a sequence of polynomials in a. We have just observed ((3.20) and remarks following) that the integral with respect to $\mu_{1,b,x}$ of any polynomial in a is an analytic element in x on $\mathbb{Z}_p$, so each integral on the right of (3.21) is an analytic element in x on $\mathbb{Z}_p$. Thus the integral on the left in (3.21) is a uniformly convergent sum on $\mathbb{Z}_p$ of analytic elements on $\mathbb{Z}_p$, and is therefore an analytic element in x on $\mathbb{Z}_p$. Putting $x = -qt$, the integral is an analytic element in t on $q^{-1}\mathbb{Z}_p$, proving the theorem. When $p = 2$, we modify the argument by observing that $\omega(a) = \lim_{r\to\infty} 1 - 2((a-1)/2)^{2^r}$ uniformly on $\mathbb{Z}_2^\times$ and $\langle a \rangle = a \cdot \omega(a)$ when $p = 2$.

4. Characters Not of p-power Conductor.

If χ is a Dirichlet character whose conductor f is not a power of p, there is no need to use a "regularized" measure to express $L_p(s, t, \chi)$ as a p-adic Γ-transform. Here again we work with $\mathfrak{O}_K$-valued measures on $\mathbb{Z}_p$, where $K = \mathbb{Q}_p(\chi)$, which are constructed directly from χ.

Theorem 4.1. *Suppose that χ is a primitive Dirichlet character whose conductor f is not a power of p. Then for all $t \in \mathbb{Z}_p$ and all $s \in \mathfrak{D}$ we have*

$$\int_{\mathbb{Z}_p^\times} \omega^m(a) \langle a \rangle^s \, d\mu(a) = -L_p(-s, t, \chi\omega^{m+1})$$

for any integer m, where $\mu = \mu_{\chi,t}$ is the $\mathfrak{O}_K$-valued measure on $\mathbb{Z}_p$ corresponding to the power *series*

$$h_{\chi,t}(T) = \sum_{a=1}^{f} \frac{\chi(a)T^{a+qt}}{T^f - 1}$$

REMARK. As in Theorem 3.1, the measure $\mu_{\chi,t}$ is an $\mathfrak{O}_K$-valued measure even for $t \in pq^{-1}\mathbb{Z}_p$, and we can therefore take these integrals as defining analytic functions of s which for $p = 2$ extend the domain of $L_2(s, t, \chi\omega^{m+1})$ to include $t \in \frac{1}{2}\mathbb{Z}_2$, which agree with the function $L_2(s, t, \chi\omega^{m+1})$ defined by Fox when $t \in \mathbb{Z}_2$. This will be further explored in §5.

Proof. In ([13], Theorem 4.1) it is verified that for any $t \in \mathbb{Z}_p$ we have $h = h_{\chi,t} \in \mathfrak{O}_K[[T-1]]$, and therefore by definition (2.1),

$$h(T) = \int_{\mathbb{Z}_p} T^a \, d\mu(a). \tag{4.1}$$

Substituting $T = e^u$ and comparing coefficients of $u^m/m!$ yields

$$a_m = \frac{B_{m+1,\chi}(qt)}{m+1} = \int_{\mathbb{Z}_p} a^m \, d\mu(a). \tag{4.2}$$

Then since

$$\psi h(T) = \sum_{a=1}^{f} \frac{\chi(ap)T^{a+p^{-1}qt}}{T^f - 1} \tag{4.3}$$

(cf. [13], eqs. (4.4), (2.15)), we obtain

$$a_m - p^m a_m^* = \frac{B_{m+1,\chi}(qt) - \chi(p)p^m B_{m+1,\chi}(p^{-1}qt)}{m+1} = \int_{\mathbb{Z}_p^\times} a^m \, d\mu_{\chi,t}(a). \tag{4.4}$$

Comparing with (1.1), we have

$$\int_{\mathbb{Z}_p^\times} a^m \, d\mu(a) = -L_p(-m, t, \chi\omega^{m+1}). \tag{4.5}$$

This says that if s is a nonnegative integer congruent to m modulo $\phi(q)$, then

$$\int_{\mathbb{Z}_p^\times} \omega^m(a)\langle a\rangle^s \, d\mu(a) = -L_p(-s, t, \chi\omega^{m+1}). \tag{4.6}$$

As the p-adic Γ-transform of the measure $\omega^m\mu$, the integral on the left is analytic for $s \in \mathfrak{D}$ ([11], Corollary 12.5); by ([7], Theorem 3.13), the L-function on the right is also analytic for $s \in \mathfrak{D}$. Since both sides are analytic and they agree on the set $\{s \in \mathbb{Z} : s > 0, s \equiv m \pmod{\phi(q)}\}$, their equality holds for all $s \in \mathfrak{D}$.

The following system of congruences, whose proof is easier than that of Corollary 3.2 due to the absence of the regularizing factor, follows directly from this theorem.

COROLLARY 4.2. *Suppose that χ is a primitive Dirichlet character whose conductor f is not a power of p. If c, k, m are any positive integers then for all $t \in \mathbb{Z}_p$,*

$$\Delta_c^k \left\{ \frac{B_{m,\chi_m}(qt) - \chi_m(p)p^{m-1}B_{m,\chi_m}(p^{-1}qt)}{m} \right\} \equiv 0 \pmod{c^k q^k \mathfrak{O}_K}$$

and

$$\binom{\rho\Delta_c}{k} \left\{ \frac{B_{m,\chi_m}(qt) - \chi_m(p)p^{m-1}B_{m,\chi_m}(p^{-1}qt)}{m} \right\} \in \mathfrak{O}_K$$

for any $\rho \in (cq)^{-1}\mathbb{Z}_p$.

For the characters involved this corollary improves Theorem 4.10 and Theorem 4.12 of [7] by relaxing the restriction on $|t|$ by a factor of $pq^{-1}F_0$, where $F_0 = \mathrm{lcm}(f, q)$, and it improves Theorem 4.2 of [13] by removing the restriction that $\phi(q)$ divides c.

By means of the isomorphism $\Lambda_{\mathfrak{O}_K} \to \mathfrak{O}_K[[T-1]]$ we can express $L_p(s, t, \chi)$ as an integral with respect to the measure $\mu_{\chi,0}$ for all $t \in \mathfrak{O}$.

COROLLARY 4.3. *If χ is a primitive Dirichlet character whose conductor f is not a power of p, then for any $t \in \mathbb{Z}_p$,*

$$\mu_{\chi,t} = \mu_{\chi,0} * \delta_{qt} \qquad \text{in} \qquad \Lambda_{\mathfrak{O}_K},$$

where δ_{qt} is the Dirac measure at qt and $*$ denotes convolution of measures. Consequently, for $t \in \mathfrak{O}$, $s \in \mathfrak{D}$, and $m \in \mathbb{Z}$ we have

$$\int_{\mathbb{Z}_p^\times} \omega^m(a)\langle a + qt\rangle^s \, d\mu_{\chi,0}(a) = -L_p(-s, t, \chi\omega^{m+1}).$$

Proof. Clearly $h_{\chi,t}(T) = h_{\chi,0}(T) \cdot T^{qt}$ in $\mathfrak{O}_K[[T-1]]$; since this ring is isomorphic to the ring $\Lambda_{\mathfrak{O}_K}$ under addition and convolution, the first statement follows by noting that under this isomorphism $h_{\chi,t}(T) \to \mu_{\chi,t}$ and $T^{qt} \to \delta_{qt}$. Therefore from Theorem 4.1, for $t \in \mathbb{Z}_p$ we have

$$-L_p(-s, t, \chi\omega^{m+1}) = \int_{\mathbb{Z}_p^\times} \omega^m(a)\langle a\rangle^s \, d(\mu_{\chi,0} * \delta_{qt})(a) = \int_{\mathbb{Z}_p^\times} \omega^m(a)\langle a + qt\rangle^s \, d\mu_{\chi,0}(a). \tag{4.7}$$

By observing that $\langle a+qt\rangle^s = \langle a\rangle^s(1+a^{-1}qt)^s$, the integral on the right can be expressed as

$$\begin{aligned}
\int_{\mathbb{Z}_p^\times} \omega^m(a)\langle a+qt\rangle^s\, d\mu_{\chi,0}(a) &= \sum_{k=0}^{\infty}\binom{s}{k}(qt)^k \int_{\mathbb{Z}_p^\times} \omega^m(a)a^{-k}\langle a\rangle^s\, d\mu_{\chi,0}(a) \\
&= \sum_{k=0}^{\infty}\binom{s}{k}(qt)^k \int_{\mathbb{Z}_p^\times} \omega^{m-k}(a)\langle a\rangle^{s-k}\, d\mu_{\chi,0}(a) \qquad (4.8) \\
&= -\sum_{k=0}^{\infty}\binom{s}{k}(qt)^k L_p(k-s,\chi\omega^{m+1-k}),
\end{aligned}$$

which clearly shows that for any $s \in \mathfrak{D}$ the integral on the right in (4.7) is analytic in t for $t \in \mathfrak{D}$. Since both sides of (4.7) are analytic for $t \in \mathfrak{D}$ and they agree for $t \in \mathbb{Z}_p$, they must be equal for all $t \in \mathfrak{D}$.

We now show how this corollary can be used to express the partial derivative with respect to s of $L_p(s,t,\chi)$ at $s=0$ in terms of p-adic special functions, generalizing formulas of Diamond [4] and Ferrero-Greenberg [6]. We recall [9] that the Iwasawa p-adic logarithm function $\log_p : \mathbb{C}_p^\times \to \mathbb{C}_p$ is the unique function on $\mathbb{C}_p^\times$ such that $\log_p(x) = \sum_{n=1}^{\infty}(-1)^n(x-1)^n/n$ for $|x-1|<1$, $\log_p(xy) = \log_p(x)+\log_p(y)$ for all $x,y \in \mathbb{C}_p^\times$, and $\log_p(p)=0$. Morita's p-adic gamma function $\Gamma_p : \mathbb{Z}_p \to \mathbb{Z}_p^\times$ is the unique continuous function on $\mathbb{Z}_p$ satisfying $\Gamma_p(0)=1$ and

$$\Gamma_p(n) = (-1)^n \prod_{\substack{0<j<n \\ p\nmid j}} j \qquad (4.9)$$

for all positive integers n (cf. [5], Ch. 21). Diamond's p-adic log gamma function G_p (not equal to $\log_p \Gamma_p$) is defined [3] for $x \in \mathbb{C}_p \setminus \mathbb{Z}_p$ by

$$G_p(x) = \lim_{r\to\infty} \sum_{j=0}^{p^r-1} (x+j)(\log_p(x+j)-1). \qquad (4.10)$$

Diamond's formula ([4], Theorem 8) for $L_p'(0,\chi)$ reads

$$L_p'(0,\chi) = \sum_{\substack{a=1 \\ p\nmid a}}^{pf-1} \chi_1(a) G_p\left(\frac{a}{pf}\right) - L_p(0,\chi)\log_p(f) \qquad (4.11)$$

for any primitive Dirichlet character χ such that χ_1 has conductor f; when $(f,p)=1$ this is equivalent to the formula

$$L_p'(0,\chi) = \sum_{a=1}^{f-1} \chi_1(a)\log_p \Gamma_p\left(\frac{a}{f}\right) - L_p(0,\chi)\log_p(f) \qquad (4.12)$$

of Ferrero and Greenberg ([6], Proposition 1). The agreement between these formulas can be established by means of the relation, for $x \in \mathbb{Z}_p$,

$$\log_p \Gamma_p(x) = \sum_{\substack{i=0 \\ x+i \in \mathbb{Z}_p^\times}}^{p-1} G_p\left(\frac{x+i}{p}\right) \tag{4.13}$$

(cf. [6]). For positive integers k, Diamond ([4], Theorem 3, Theorem 5) also showed that

$$L_p(k, \chi_k) = \frac{(-pf)^{-k}}{(k-1)!} \sum_{\substack{a=1 \\ p \nmid a}}^{pf-1} \chi_1(a) G_p^{(k)}\left(\frac{a}{pf}\right) \tag{4.14}$$

(with the obvious exception that $k \neq 1$ if $\chi_1 = 1$); again, if the conductor f of χ_1 is coprime to p this is equivalent by (4.13) to

$$L_p(k, \chi_k) = \frac{(-f)^{-k}}{(k-1)!} \sum_{a=1}^{f-1} \chi_1(a) (\log_p \Gamma_p)^{(k)}\left(\frac{a}{f}\right). \tag{4.15}$$

Theorem 4.4. *If χ is a primitive Dirichlet character such that the conductor f of χ_1 is not a power of p, then for any $t \in \mathfrak{O}$,*

$$\frac{\partial}{\partial s} L_p(0, t, \chi) = \sum_{\substack{a=1 \\ p \nmid a}}^{pf-1} \chi_1(a) G_p\left(\frac{a+qt}{pf}\right) - L_p(0, \chi) \log_p(f).$$

If $(f, p) = 1$ and $t \in \mathbb{Z}_p$ this is equivalent to

$$\frac{\partial}{\partial s} L_p(0, t, \chi) = \sum_{a=1}^{f-1} \chi_1(a) \log_p \Gamma_p\left(\frac{a+qt}{f}\right) - L_p(0, \chi) \log_p(f).$$

Proof. By Corollary 4.3,

$$\begin{aligned} \frac{\partial}{\partial s} L_p(s, t, \chi) &= -\frac{\partial}{\partial s} \int_{\mathbb{Z}_p^\times} \omega^{-1}(a) \langle a+qt \rangle^{-s} \, d\mu(a) \\ &= \int_{\mathbb{Z}_p^\times} \omega^{-1}(a) \langle a+qt \rangle^{-s} \log_p \langle a+qt \rangle \, d\mu(a), \end{aligned} \tag{4.16}$$

where $\mu = \mu_{\chi,0}$, which shows that in particular

$$\frac{\partial}{\partial s} L_p(0, t, \chi) = \int_{\mathbb{Z}_p^\times} \omega^{-1}(a) \log_p \langle a+qt \rangle \, d\mu(a). \tag{4.17}$$

Since $\langle a+qt\rangle = \langle a\rangle(1+a^{-1}qt)$ for all $a \in \mathbb{Z}_p^\times$ and all $t \in \mathfrak{O}$, by Theorem 4.1, (4.11), and (4.14) we have

$$
\begin{aligned}
\frac{\partial}{\partial s} L_p(0,t,\chi) &= \int_{\mathbb{Z}_p^\times} \omega^{-1}(a)\log_p\langle a\rangle\, d\mu(a) + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}(qt)^k \int_{\mathbb{Z}_p^\times} \omega^{-1}(a)a^{-k}\, d\mu(a) \\
&= L_p'(0,\chi) + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}(qt)^k \int_{\mathbb{Z}_p^\times} \omega^{-k-1}(a)\langle a\rangle^{-k}\, d\mu(a) \\
&= L_p'(0,\chi) + \sum_{k=1}^{\infty} \frac{(-1)^{k}}{k}(qt)^k L_p(k,\chi_k) \\
&= \sum_{\substack{a=1 \\ p\nmid a}}^{pf-1} \chi_1(a) G_p\left(\frac{a}{pf}\right) - L_p(0,\chi)\log_p(f) \\
&\qquad + \sum_{k=1}^{\infty} \frac{(-1)^{k}}{k}(qt)^k \left(\frac{(-pf)^{-k}}{(k-1)!} \sum_{\substack{a=1 \\ p\nmid a}}^{pf-1} \chi_1(a) G_p^{(k)}\left(\frac{a}{pf}\right)\right) \\
&= \sum_{\substack{a=1 \\ p\nmid a}}^{pf-1} \chi_1(a) \sum_{k=0}^{\infty} \frac{G_p^{(k)}(\frac{a}{pf})}{k!}\left(\frac{qt}{pf}\right)^k - L_p(0,\chi)\log_p(f) \\
&= \sum_{\substack{a=1 \\ p\nmid a}}^{pf-1} \chi_1(a) G_p\left(\frac{a+qt}{pf}\right) - L_p(0,\chi)\log_p(f),
\end{aligned}
\tag{4.18}
$$

using the fact that G_p is locally analytic on $\mathbb{C}_p \setminus \mathbb{Z}_p$ (cf. [3]). The second statement then follows from (4.13).

In ([7], Theorem 4.5), Fox proved that for all characters χ, the L-function has an expansion

$$
L_p(s,t,\chi) = \sum_{m=0}^{\infty} \binom{-s}{m} (qt)^m L_p(s+m,\chi_m) \tag{4.19}
$$

valid for $t \in \mathfrak{O}$ and $s \in \mathfrak{D}$. (We derived this formula for characters χ whose conductor is not a power of p in (4.7), (4.8)). This result directly implies a generalization of Diamond's formula (4.14).

Theorem 4.5. *Let χ be a primitive Dirichlet character such that χ_1 has conductor f, and let k be any positive integer. Then, with the obvious exception that $k \neq 1$ if $\chi_1 = 1$, for all $s \in \mathfrak{D}$ and $t \in \mathfrak{O}$ we have*

$$
L_p(k,t,\chi_k) = \frac{(-pf)^{-k}}{(k-1)!} \sum_{\substack{a=1 \\ p\nmid a}}^{pf-1} \chi_1(a) G_p^{(k)}\left(\frac{a+qt}{pf}\right).
$$

If $(f,p)=1$ and $t\in\mathbb{Z}_p$ this is equivalent to

$$L_p(k,t,\chi_k)=\frac{(-f)^{-k}}{(k-1)!}\sum_{a=1}^{f-1}\chi_1(a)(\log_p\Gamma_p)^{(k)}\left(\frac{a+qt}{f}\right).$$

Proof. Using (4.19) and (4.14) we compute

$$\begin{aligned}
L_p(k,t,\chi_k)&=\sum_{m=0}^{\infty}\binom{-k}{m}(qt)^m L_p(k+m,\chi_{m+k})\\
&=\sum_{m=0}^{\infty}\binom{-k}{m}(qt)^m\frac{(-pf)^{-k-m}}{(k+m-1)!}\sum_{\substack{a=1\\ p\nmid a}}^{pf-1}\chi_1(a)G_p^{(k+m)}\left(\frac{a}{pf}\right)\\
&=\frac{(-pf)^{-k}}{(k-1)!}\sum_{\substack{a=1\\ p\nmid a}}^{pf-1}\chi_1(a)\sum_{m=0}^{\infty}\left(\frac{qt}{pf}\right)^m\frac{G_p^{(k+m)}\left(\frac{a}{pf}\right)}{m!}\\
&=\frac{(-pf)^{-k}}{(k-1)!}\sum_{\substack{a=1\\ p\nmid a}}^{pf-1}\chi_1(a)G_p^{(k)}\left(\frac{a+qt}{pf}\right),
\end{aligned}\tag{4.20}$$

using the local analyticity of $G_p^{(k)}$ on $\mathbb{C}_p\setminus\mathbb{Z}_p$. The second statement then follows from (4.13).

5. Extending $L_p(s,t,\chi)$ When $p=2$.

In this last section we describe how $L_2(s,t,\chi)$ may be extended via the 2-adic Γ-transform to $t\in\frac{1}{2}\mathbb{Z}_2$, when the character χ is not of the second kind. The extension is not unique, as it is analytic in t only on $\mathfrak{D}$, but is sufficient to imply congruences which complement those of Corollary 3.2 and Corollary 4.2 of this paper, and generalize Theorem 4.2 of [13]. We say that a Dirichlet character χ is a character *of the first kind* if either $f=d$ or $f=dq$ with $(d,p)=1$; we say that χ is a character *of the second kind* if either $f=1$ or $f=qp^e$ with $e\geq 1$.

Theorem 5.1. *Let $p=2$, and let χ be a Dirichlet character which is not of the second kind. Then there exists a function $L_2(s,t,\chi)$ which is an analytic function of s on $\mathfrak{D}$ for each $t\in\frac{1}{2}\mathbb{Z}_2$, and is an analytic element in t on $\frac{1}{2}\mathbb{Z}_2$ for all $s\in\mathfrak{D}$, and such that for any positive integer m,*

$$L_2(1-m,t,\chi)=-\frac{B_{m,\chi_m}(4t)-\chi_m(2)2^{m-1}B_{m,\chi_m}(2t)}{m}$$

for all $t\in\mathbb{Z}_2$, and

$$L_2(1-m,t,\chi)=(-1)^m\frac{B_{m,\chi_m}(4t)-\chi_m(2)2^{m-1}B_{m,\chi_m}(2t)}{m}$$

for all $t \in \frac{1}{2}\mathbb{Z}_2^\times$.

Proof. Suppose first that $\chi = \omega$, and define $L_2(s,t,\omega)$, as in Theorems 3.1 and 3.3, by

$$\int_{\mathbb{Z}_2^\times} \langle a\rangle^s \, d\mu_{1,b,x}(a) = (1-\omega(b)\langle b\rangle^{s+1})L_2(-s,t,\omega), \tag{5.1}$$

where b is a fixed positive integer congruent to -1 modulo 4, and $x = -4t$. Then $L_2(s,t,\omega)$ is analytic for $s \in \mathfrak{D}$ for any $t \in \frac{1}{2}\mathbb{Z}_2$, and for $t \in \mathbb{Z}_2$ has the indicated value at $s = 1-m$ for positive integers m. Furthermore, since $\langle a\rangle^s = a^s$ for all $a \in \mathbb{Z}_2^\times$ if s is an even positive integer, by (3.7) the function $L_2(s,t,\omega^i)$ also has the indicated value at $s = 1-m$ when $t \in \frac{1}{2}\mathbb{Z}_2^\times$ and m is odd. So to complete the verification of the formula in this case we must compute $L_2(1-m,t,\omega)$ when $t \in \frac{1}{2}\mathbb{Z}_2^\times$ and m is even.

Using the identity $\omega(a) = (-i/2)i^a + (i/2)(-i)^a$ for all $a \in \mathbb{Z}_2$, where i denotes a fixed square root of -1, we have

$$\int_{\mathbb{Z}_2} \omega(a)T^a \, d\mu_{1,b,x}(a) = (-i/2)h_{1,b,x}(iT) + (i/2)h_{1,b,x}(-iT). \tag{5.2}$$

Now we assume $t \in \frac{1}{2}\mathbb{Z}_2^\times$, so that $x = -4t \in 2\mathbb{Z}_2 \setminus 4\mathbb{Z}_2$; therefore $\omega(a-x) = -\omega(a)$ for any $a \in \mathbb{Z}_2$. Then writing

$$\begin{aligned} h_{1,b,x}(T) &= \frac{bT^{b(1-x)}}{T^b-1} - \frac{T^{1-x}}{T-1} \\ &= b\sum_{a=1}^{4} \frac{T^{b(a-x)}}{T^{4b}-1} - \sum_{a=1}^{4}\frac{T^{a-x}}{T^4-1}, \end{aligned} \tag{5.3}$$

we find

$$\begin{aligned} \int_{\mathbb{Z}_2} \omega(a)T^a \, d\mu_{1,b,x}(a) &= (-i/2)h_{1,b,x}(iT) + (i/2)h_{1,b,x}(-iT) \\ &= b\sum_{a=1}^{4} \frac{\omega(b(a-x))T^{b(a-x)}}{T^{4b}-1} - \sum_{a=1}^{4}\frac{\omega(a-x)T^{a-x}}{T^4-1} \\ &= -b\omega(b)\sum_{a=1}^{4} \frac{\omega(a)T^{b(a-x)}}{T^{4b}-1} + \sum_{a=1}^{4}\frac{\omega(a)T^{a-x}}{T^4-1} \\ &= -h_{\omega,b,x}(T). \end{aligned} \tag{5.4}$$

As observed in Theorem 3.1, $\psi h_{\omega,b,x} = 0$ because $x \in 2\mathbb{Z}_2$; therefore,

$$\begin{aligned} \int_{\mathbb{Z}_2} \omega(a)T^a \, d\mu_{1,b,x}(a) &= \int_{\mathbb{Z}_2^\times} \omega(a)T^a \, d\mu_{1,b,x}(a) \\ &= -\int_{\mathbb{Z}_2^\times} T^a \, d\mu_{\omega,b,x}(a), \end{aligned} \tag{5.5}$$

and substituting $T = e^u$ in (5.5) and comparing coefficients of $u^m/m!$ gives

$$\int_{\mathbb{Z}_2^\times} \omega(a) a^m \, d\mu_{1,b,x}(a) = -\int_{\mathbb{Z}_2^\times} a^m \, d\mu_{\omega,b,x}(a) = -(b^{m+1}\omega(b) - 1)\frac{B_{m+1,\omega}(-x)}{m} \tag{5.6}$$

for any positive integer m. Since $\omega(a)a^m = \langle a\rangle^m$ and $b^{m+1} = \langle b\rangle^{m+1}$ when m is an odd integer, (5.6) says that

$$\begin{aligned}\int_{\mathbb{Z}_2^\times} \langle a\rangle^m \, d\mu_{1,b,x}(a) &= -\int_{\mathbb{Z}_2^\times} a^m \, d\mu_{\omega,b,x}(a) \\ &= (-1)^{m+1}(1 - \chi(b)\langle b\rangle^{m+1})\frac{B_{m+1,\chi_{m+1}}(4t) - \chi_{m+1}(2)2^m B_{m+1,\chi_{m+1}}(2t)}{m}\end{aligned} \tag{5.7}$$

when $\chi = \omega$, $x = -4t \in 2\mathbb{Z}_2 \setminus 4\mathbb{Z}_2$, and m is an odd positive integer. This proves the formula for $\chi = \omega$.

Now if χ is a primitive Dirichlet character whose conductor f is not a power of 2, then we define

$$\int_{\mathbb{Z}_2^\times} \langle a\rangle^s \, d\mu_{\chi\omega,t}(a) = -L_2(-s, t, \chi), \tag{5.8}$$

for $t \in \frac{1}{2}\mathbb{Z}_2$, with $\mu_{\chi\omega,t}$ as in Theorem 4.1. As in the case just considered, $L_2(s,t,\chi)$ is analytic for $s \in \mathfrak{D}$ for any such t, and has the indicated value at $s = 1-m$ whenever $t \in \mathbb{Z}_2$ or when m is odd. By definition of $\mu_{\chi\omega,t}$, we have

$$\int_{\mathbb{Z}_2} T^a \, d\mu_{\chi\omega,t}(a) = h_{\chi\omega,t}(T) = \sum_{a=1}^{f} \frac{\chi\omega(a)T^{a+qt}}{T^f - 1} = \sum_{a=1}^{4f} \frac{\chi\omega(a)T^{a+qt}}{T^{4f} - 1}, \tag{5.9}$$

where f is the conductor of $\chi\omega$. Then since $\omega(a) = (-i/2)i^a + (i/2)(-i)^a$ for all $a \in \mathbb{Z}_2$, we get

$$\int_{\mathbb{Z}_2} \omega(a) T^a \, d\mu_{\chi\omega,t}(a) = -\sum_{a=1}^{4f} \frac{\chi\omega(a)\omega(a)T^{a+qt}}{T^{4f} - 1} \tag{5.10}$$

for $t \in \frac{1}{2}\mathbb{Z}_2^\times$, invoking the identity $\omega(a+qt) = -\omega(a)$ for such t. Applying the φ operator to both sides of (5.10) gives

$$\int_{\mathbb{Z}_2^\times} \omega(a) T^a \, d\mu_{\chi\omega,t}(a) = -\sum_{\substack{a=1 \\ a \text{ odd}}}^{4f} \frac{\chi\omega(a)\omega(a)T^{a+qt}}{T^{4f} - 1} = -\varphi h_{\chi,t}(T), \tag{5.11}$$

since $\chi\omega(a)\omega(a) = \chi(a)$ for odd integers a. Substituting $T = e^u$ and comparing coefficients of $u^m/m!$ gives

$$\int_{\mathbb{Z}_2^\times} \omega(a) a^m \, d\mu_{\chi\omega,t}(a) = -\int_{\mathbb{Z}_2^\times} a^m \, d\mu_{\chi,t}(a) = -\frac{B_{m+1,\chi}(4t) - \chi(2)2^m B_{m+1,\chi}(2t)}{m} \tag{5.12}$$

for any positive integer m when $t \in \frac{1}{2}\mathbb{Z}_2^\times$. So if m is odd we have

$$\int_{\mathbb{Z}_2^\times} \langle a\rangle^m \, d\mu_{\chi\omega,t}(a) = (-1)^m \frac{B_{m+1,\chi_{m+1}}(4t) - \chi_{m+1}(2)2^m B_{m+1,\chi_{m+1}}(2t)}{m}, \tag{5.13}$$

proving the formula.

In either case, $L_2(s,t,\omega)$ can be shown to be an analytic element in t on $\frac{1}{2}\mathbb{Z}_2$ by the same argument used in the proof of Theorem 3.3. This completes the proof of this theorem.

It will be observed that the function described in this theorem is not canonical; indeed, instead of (5.1) we could have defined a function $\tilde{L}_2(s,t,\omega)$ by

$$\int_{\mathbb{Z}_2^\times} \omega(a)\langle a\rangle^s \, d\mu_{\omega,b,x}(a) = (1 - \omega(b)\langle b\rangle^{s+1})\tilde{L}_2(-s,t,\omega), \tag{5.14}$$

and instead of (5.8) we could have defined

$$\int_{\mathbb{Z}_2^\times} \omega(a)\langle a\rangle^s \, d\mu_{\chi,t}(a) = -\tilde{L}_2(-s,t,\chi) \tag{5.15}$$

for characters χ whose conductor is not a power of 2. While these agree with (5.1), (5.8) for $t \in \mathbb{Z}_2$, they differ from (5.1), (5.8) in sign for $t \in \frac{1}{2}\mathbb{Z}_2^\times$. That is, $L_2(s,t,\chi) = \tilde{L}_2(s,t,\chi)$ when $t \in \mathbb{Z}_2$ but $L_2(s,t,\chi) = -\tilde{L}_2(s,t,\chi)$ when $t \in \frac{1}{2}\mathbb{Z}_2^\times$. The definition (5.1), however, coincides with Theorem 3.3 wherein $L_2(s,t,\omega)$ was defined on $\frac{1}{4}\mathbb{Z}_2$.

We conclude by stating the congruences for the generalized Bernoulli polynomials which follow directly from this theorem.

Corollary 5.2. *Let $p = 2$, let χ be a Dirichlet character which is not of the second kind, and let $t \in \frac{1}{2}\mathbb{Z}_2^\times$. Then for any positive integers c, k, and m,*

$$\Delta_c^k \left\{ (-1)^m \frac{B_{m,\chi_m}(4t) - \chi_m(2)2^{m-1}B_{m,\chi_m}(2t)}{m} \right\} \equiv 0 \pmod{\frac{1}{2}4^k c^k \mathfrak{D}_K}$$

and

$$\binom{\rho\Delta_c}{k} \left\{ (-1)^m \frac{B_{m,\chi_m}(4t) - \chi_m(2)2^{m-1}B_{m,\chi_m}(2t)}{m} \right\} \in \frac{1}{2}\mathfrak{D}_K$$

for any $\rho \in (4c)^{-1}\mathbb{Z}_2$.

When c is even, this result is covered by ([13], Theorem 4.2); indeed the factor of $(-1)^m$ in the terms of the congruences was invisible to that theorem because of its assumption that c was even. When $\chi \neq \omega$ the factor of $1/2$ in the moduli of the congruences may be omitted; this factor arises from the 2-ordinal of the regularizing factor, as in the proof of Corollary 3.2.

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