A 2-adic formula for Bernoulli numbers of the second kind and for the Nörlund numbers

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Abstract

We give a formula expressing Bernoulli numbers of the second kind as 2-adically convergent sums of traces of algebraic integers. We use this formula to prove and explain the formulas and conjectures of Adelberg concerning the initial 2-adic digits of these numbers. We also give analogous results for the Nörlund numbers.

Keywords: Bernoulli numbers of second kind, Nörlund numbers, Cauchy numbers, congruences, p-adic analysis

1. Introduction

The *Bernoulli numbers of the second kind* b_n are the rational numbers determined ([2], [5]) by the generating function

$$\frac{t}{\log(1+t)} = \sum_{n=0}^{\infty} b_n t^n. \tag{1.1}$$

The numbers $n!b_n$ have also been called *Cauchy numbers of the first type* ([3], [7]), and may be defined by

$$n!b_n = \int_0^1 x(x-1)(x-2)\cdots(x-n+1)\,dx. \tag{1.2}$$

The first few values are $b_0 = 1$, $b_1 = 1/2$, $b_2 = -1/12$, $b_3 = 1/24$, $b_4 = -19/720$, $b_5 = 3/160$. Their applications in number theory include

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{b_n}{n} = \gamma; \qquad 1 + \sum_{n=1}^{\infty}(-1)^{n+1}\frac{b_n H_n}{n} = \frac{\pi^2}{6}, \tag{1.3}$$

(cf. [7], §4) where $H_n = \sum_{k=1}^{n} 1/k$ is the n-th harmonic number and $\gamma = \lim_{n\to\infty}(H_n - \log n)$ denotes Euler's constant. Among the interesting arithmetic properties of these numbers are the large initial gaps in their 2-adic expansions; for example,

$$2^{15}b_{15} \equiv -1 + 2^8 + 2^{16} \pmod{2^{17}\mathbb{Z}_2}, \tag{1.4}$$

$$2^{20}b_{20} \equiv 1 + 2^{18} \pmod{2^{19}\mathbb{Z}_2}, \tag{1.5}$$

$$2^{30}b_{30} \equiv 1 - 2^{16} + 2^{33} \pmod{2^{34}\mathbb{Z}_2}. \tag{1.6}$$

This property was studied recently by Adelberg [1], who proved that $(-2)^n b_n \equiv 1 \pmod{2^{[n/2]+1}\mathbb{Z}_2}$ where $[x]$ is the greatest integer function, and showed that this congruence is best possible if 4 does not divide n; this essentially determines the first gap. He also made several conjectures concerning the second gap on the basis of numerical computation. In this note we give a simple 2-adic formula for the numbers b_n and use it to verify those conjectures. In fact we prove the following:

Theorem 2. *For all nonnegative integers n, we have*

$$(-2)^n b_n \equiv 1 + \varepsilon_n 2^{[n/2]+1} + 2^{e_n} \qquad (\text{mod } 2^{e_n+1}\mathbb{Z}_2)$$

where ε_n is given by

$$\varepsilon_n = \begin{cases} 1, & \text{if } n \equiv 1,2,3 \text{ (mod 8)}, \\ -1, & \text{if } n \equiv 5,6,7 \text{ (mod 8)}, \\ 0, & \text{if } n \equiv 0,4 \text{ (mod 8)}, \end{cases}$$

and e_n is given by

n	$8k-2$	$8k-1$	$8k$	$8k+1$	$8k+2$	$8k+3$	$8k+4$	$8k+5$
e_n	$7k+\mathrm{ord}8k$	$6k+\mathrm{ord}8k$	$6k+\mathrm{ord}16k$	$6k+2$	$6k+4$	$6k+4$	$6k+6$	$6k+5$,

where ord *is the 2-adic valuation normalized by* $\mathrm{ord}2 = 1$.

This theorem gives a simple formula which precisely predicts at least the first $\frac{3n}{4}$ 2-adic digits of any b_n. The values of ε_n were called the *conjectured stable congruences* for the b_n by Adelberg ([1], p. 57), who also observed the above values of e_n through numerical computation, although he declined to state a formula for e_n in the cases $n \equiv 0, -1, -2 \pmod 8$ where the formula involves $\mathrm{ord}k$. Although the calculation of e_n in these three cases requires slightly more delicate analysis than the others, these congruences all follow by essentially the same line of argument from the following formula:

Theorem 1. *For each $r \geq 0$ let ζ_r denote any primitive 2^{r+1}-th root of unity. Then for all nonnegative integers n we have*

$$(-2)^n b_n = -\sum_{r=0}^{\infty} \mathrm{Tr}_r \left(\zeta_r \left(\frac{2}{1-\zeta_r} \right)^n \right)$$

as a 2-adically convergent sum of integers, where Tr_r denotes the trace map from $\mathbb{Q}(\zeta_r)$ to $\mathbb{Q}$.

The *Bernoulli numbers of order w*, $B_n^{(w)}$, are the rational numbers defined [8] by the generating function

$$\left(\frac{t}{e^t - 1} \right)^w = \sum_{n=0}^{\infty} B_n^{(w)} \frac{t^n}{n!}. \tag{1.7}$$

For $n = w$ the numbers $B_n^{(n)}$ are called *Nörlund numbers* [4], or *Cauchy numbers of the second type* ([3], [7]), and may be determined by the generating function

$$\frac{t}{(1+t)\log(1+t)} = \sum_{n=0}^{\infty} B_n^{(n)} \frac{t^n}{n!} \tag{1.8}$$

(cf. [8]). The first few values are $B_0^{(0)} = 1$, $B_1^{(1)} = -1/2$, $B_2^{(2)} = 5/6$, $B_3^{(3)} = -9/4$, $B_4^{(4)} = 251/30$, $B_5^{(5)} = -475/12$. One important role they play in combinatorial analysis is through the formula

$$B_n^{(n)} = \int_0^1 (x-1)(x-2)\cdots(x-n)\, dx \tag{1.9}$$

(cf. [8]). They are related to the Bernoulli numbers of the second kind via the formulas [4]

$$\frac{B_n^{(n)}}{n!} = \sum_{j=0}^{n} (-1)^{n-j} b_j \qquad \text{and} \qquad b_n = \frac{B_n^{(n)}}{n!} + \frac{B_{n-1}^{(n-1)}}{(n-1)!}. \tag{1.10}$$

Adelberg also studied their 2-adic expansion in [1] and proved that $(-2)^n B_n^{(n)}/n! \equiv -1 \pmod{2^{[n/2]+1}\mathbb{Z}_2}$, with this congruence being best possible if $n \not\equiv 2 \pmod 4$ ([1], Theorem 2 and Corollary 1 of §3); he further conjectured "stable congruences" for them which were supported by numerical calculations. In §4 we give a formula similar to Theorem 1 for $B_n^{(n)}/n!$ which furnishes proof of Adelberg's conjectures concerning the 2-adic digits of these numbers.

2. Proof of 2-adic formula

Throughout this paper $\mathbb{Z}_2$ will denote the ring of 2-adic integers and $\mathbb{Q}_2$ will denote the field of 2-adic numbers. Clearly $\zeta = \zeta_r$ is a primitive 2^{r+1}-th root of unity if and only if $\zeta^{2^r} = -1$, so the minimal polynomial for ζ_r over $\mathbb{Q}$ is the 2^{r+1}-th cyclotomic polynomial $\Phi_{2^{r+1}}(t) = t^{2^r} + 1$. It is well known that $\mathbb{Q}(\zeta_r)$ is an abelian extension of $\mathbb{Q}$ with Galois group $\mathrm{Gal}(\mathbb{Q}(\zeta_r)/\mathbb{Q}) = \{\sigma_j : j \in (\mathbb{Z}/2^{r+1}\mathbb{Z})^\times\} \cong (\mathbb{Z}/2^{r+1}\mathbb{Z})^\times$, where σ_j is the automorphism of $\mathbb{Q}(\zeta_r)$ induced by $\zeta_r \mapsto \zeta_r^j$. Since $1-\zeta_r$ is a root of the 2-Eisenstein polynomial $\Phi_{2^{r+1}}(1-t) = 2 - 2^r t + \cdots - 2^r t^{2^r-1} + t^{2^r}$ we have $\mathrm{ord}(1-\zeta_r) = 1/2^r$; hence the degree-$2^r$ extension $K_r = \mathbb{Q}_2(\zeta_r)$ of $\mathbb{Q}_2$ is totally ramified, with ring of integers $\mathfrak{O}_r = \{x \in K_r : \mathrm{ord} x \geq 0\}$, maximal ideal $\mathfrak{P}_r = \{x \in K_r : \mathrm{ord} x \geq 1/2^r\}$, and residue class field $\mathfrak{O}_r/\mathfrak{P}_r$ isomorphic to $\mathbb{Z}/2\mathbb{Z}$ (cf. [6]). It follows that for $x, y \in K_r$ we have $\mathrm{ord}(x+y) \geq \min\{\mathrm{ord} x, \mathrm{ord} y\}$ with equality *if and only if* $\mathrm{ord} x \neq \mathrm{ord} y$.

Proof of Theorem 1. For any $r \geq 0$,

$$\frac{-2^{r+2}t}{(1-2t)^{2^{r+1}}-1} - \frac{-2^{r+1}t}{(1-2t)^{2^r}-1} = \frac{2^{r+1}t}{(1-2t)^{2^r}+1}. \tag{2.1}$$

If we sum this equation from $r=1$ to $r=s$, the left side telescopes, yielding

$$\frac{-2^{s+2}t}{(1-2t)^{2^{s+1}}-1} - \frac{1}{1-t} = \sum_{r=1}^{s} \frac{2^{r+1}t}{(1-2t)^{2^r}+1}. \tag{2.2}$$

Since $((1-2t)^{2^r}+1)/2$ is a unit in the power series ring $\mathbb{Z}_2[[t]]$, the r-th term in the sum in (2.2) lies in $2^r\mathbb{Z}_2[[t]]$, hence the 2-adic limit of partial sums as $s \to \infty$ exists. Since $((1+t)^a-1)/a \to \log(1+t)$ as $a \to 0$ 2-adically, we have

$$\frac{-2t}{\log(1-2t)} - \frac{1}{1-t} = \sum_{r=1}^{\infty} \frac{2^{r+1}t}{(1-2t)^{2^r}+1} \tag{2.3}$$

as an identity in $\mathbb{Z}_2[[t]]$. Expanding the left side of (2.3) as a power series gives

$$\sum_{n=0}^{\infty} c_n t^n = \sum_{r=1}^{\infty} \frac{2^{r+1}t}{(1-2t)^{2^r}+1} \tag{2.4}$$

where $c_n = (-2)^n b_n - 1$ for all n, as in [1]. We define rational integers $c_{r,n}$ by

$$\sum_{n=0}^{\infty} c_{r,n} t^n = \frac{2^{r+1}t}{(1-2t)^{2^r}+1} \tag{2.5}$$

so that $c_n = \sum_{r=1}^{\infty} c_{r,n}$ as a convergent sum in $\mathbb{Z}_2$ for all $n \geq 0$.

If $Q(t) = \prod_{i=1}^{n}(1-\alpha_i t)$ is a polynomial of degree n with distinct roots and $P(t)$ is a polynomial of degree less than n, it is easily seen that there is a partial fraction decomposition of $P(t)/Q(t)$ as $\sum_{i=1}^{n} a_i/(1-\alpha_i t)$ where $a_i = -\alpha_i P(\alpha_i^{-1})/Q'(\alpha_i^{-1})$ for all i. Thus since $(1-2t)^{2^r}+1=0$ whenever $2t = 1-\zeta_r$ for a primitive 2^{r+1}-th root of unity ζ_r, we have for each r by partial fraction decomposition

$$\sum_{n=0}^{\infty} c_{r,n} t^n = \frac{2^{r+1}t}{(1-2t)^{2^r}+1} = \sum_{\zeta} \frac{-\zeta}{1-\alpha t} = \sum_{n=0}^{\infty} \sum_{\zeta} -\zeta\alpha^n t^n \tag{2.6}$$

where $\alpha = 2/(1-\zeta)$ and the sums indexed by ζ are taken over *all* primitive 2^{r+1}-th roots of unity $\zeta = \zeta_r$. Therefore

$$c_{r,n} = -\sum_{\zeta} \zeta \left(\frac{2}{1-\zeta}\right)^n = -\mathrm{Tr}_r\left(\zeta_r\left(\frac{2}{1-\zeta_r}\right)^n\right) \tag{2.7}$$

for all r, n, where each ζ_r denotes any fixed primitive 2^{r+1}-th root of unity. This completes the proof.

Remarks. One may use this method to derive similar p-adic formulas for b_n for any prime p. Heuristically the proof of this formula gives

$$(-2)^n b_n = 1 + \sum_{r=1}^{\infty} c_{r,n} = -\sum \zeta \left(\frac{2}{1-\zeta}\right)^n \tag{2.8}$$

where the latter sum is over all nontrivial roots of unity ζ of 2-power order; however in this form the latter sum is 2-adically divergent, since the terms are bounded away from zero. By grouping terms together with their Galois conjugates, the trace maps combine the terms in (2.8) to produce the 2-adically convergent sum of the theorem. A consequence of this theorem is that for each n the sequence of traces $\{\mathrm{Tr}_r(\zeta_r(2/(1-\zeta_r))^n)\}$ converges 2-adically to zero as $r \to \infty$, which is a bit unexpected in light of the fact that $\{\zeta_r(2/(1-\zeta_r))^n\}$ is 2-adically divergent as $r \to \infty$ for every n.

The dependence of ε_n and e_n on the residue class of n modulo 8 will be explained more or less by this expression of c_n as a 2-adically convergent sum of the linearly recurrent sequences $\{c_{r,n}\}_{n=0}^{\infty}$. From (2.5) we see that for each r the sequence $\{c_{r,n}\}$ satisfies a linear recurrence of order 2^r, the reciprocal roots α of whose characteristic polynomial all have 2-adic ordinal $(2^r-1)/2^r$.

Lemma. *With $c_{r,n}$ as defined in (2.5) we have*

$$\mathrm{ord} c_{r,n} \geq \left\lceil r + (n-1)\frac{2^r-1}{2^r} \right\rceil$$

for all positive integers r and n, where $\lceil x \rceil$ denotes the least integer not less than x.

Proof. Write the characteristic polynomial $P_r(t) = ((1-2t)^{2^r}+1)/2 = 1-2^r t + \cdots + 2^{2^r-1}t^{2^r} \in \mathbb{Z}[t]$. If we introduce a change of variables $u = 2t/(1-\zeta_r)$ then $P_r(t) \in \mathfrak{O}_r[u]$ with constant term 1, so that $P_r(t)$ is a unit in $\mathfrak{O}_r[[u]]$. It follows that $P_r(t)^{-1} = \sum a_{r,n}t^n \in \mathfrak{O}_r[[t]]$ with $\mathrm{ord} a_{r,n} \geq n(2^r-1)/2^r$. Since $c_{r,n} = 2^r a_{r,n-1} \in \mathbb{Z}$ the statement of the lemma follows.

Remark. Adelberg's theorem ([1], Theorem 2) that $(-2)^n b_n \equiv 1 \pmod{2^{[n/2]+1}\mathbb{Z}_2}$ follows immediately from this lemma since $[n/2]+1 = \lceil 1 + (n-1)/2 \rceil$.

3. Proof of 2-adic congruences

In order to prove Theorem 2 we write

$$c_n = c_{1,n} + c_{2,n} + c_{3,n} + \sum_{r=4}^{\infty} c_{r,n} \tag{3.1}$$

where $c_n = (-2)^n b_n - 1$ and $c_{r,n}$ is as defined in (2.5). Theorem 2 then follows from the following three assertions:

(1). $c_{1,n} = \varepsilon_n 2^{[n/2]+1}$ for all n;

(2). $\operatorname{ord}(c_{2,n} + c_{3,n}) = e_n$ for all n;

(3). $\operatorname{ord} c_{r,n} > e_n$ for all $r \geq 4$ and all n.

A straightforward calculation from the $r = 1$ case of (2.5) or (2.7) yields

$$c_{1,n} = i((1-i)^n - (1+i)^n) = \begin{cases} 0, & \text{if } n = 4k, \\ 2(-4)^k, & \text{if } n = 4k+1, \\ -(-4)^{k+1}, & \text{if } n = 4k+2, \\ -(-4)^{k+1}, & \text{if } n = 4k+3, \end{cases} \tag{3.2}$$

where $i^2 = -1$. Assertion (1) follows directly; from this calculation and the $r \geq 2$ cases of the lemma we see that Adelberg's congruence $(-2)^n b_n \equiv 1 \pmod{2^{[n/2]+1}\mathbb{Z}_2}$ is indeed best possible in all cases except when $4|n$ ([1], §3, Corollary 2). Indeed the lemma already proves that $(-2)^n b_n \equiv 1 + \varepsilon_n 2^{[n/2]+1} \pmod{2^{\lceil (3n+5)/4 \rceil}\mathbb{Z}_2}$, which contains the "conjectured stable congruences" of Adelberg ([1], p. 57). The remainder of this section is devoted to the calculation of the exact modulus e_n.

If $\zeta = \zeta_r$ denotes a primitive 2^{r+1}-th root of unity and $\alpha = 2/(1-\zeta)$, then $-\alpha\zeta = \overline{\alpha}$ and therefore $\alpha^2(-\zeta) = |\alpha|^2$ (where $\overline{\alpha}$ denotes complex conjugate and $|\alpha|$ denotes complex absolute value). It follows that $\omega = \alpha/|\alpha|$ is a primitive 2^{r+2}-th root of unity and in fact $\omega^{-2} = -\zeta$. We may therefore rewrite (2.7) as

$$c_{r,n} = -\operatorname{Tr}_r(\zeta\alpha^n) = -\sum_{\zeta} \zeta\alpha^n = \sum_{\alpha} |\alpha|^n \omega^{n-2} \tag{3.3}$$

where the latter sum is over all 2^r values of $\alpha = 2/(1-\zeta)$ for primitive 2^{r+1}-th roots of unity ζ, with $\omega = \alpha/|\alpha|$. By pairing each such α with its complex conjugate we may write

$$c_{r,n} = \sum_{\alpha} |\alpha|^n (\omega^{n-2} + \overline{\omega}^{n-2}) \tag{3.4}$$

where the sum is now over all 2^{r-1} such values of α with positive imaginary part.

We mention two immediate consequences of this version of the formula. First, it directly implies that $c_{r,n} = 0$ when $n \equiv 2^r + 2 \pmod{2^{r+1}}$, because then each factor $\omega^{n-2} + \overline{\omega}^{n-2} = 0$. More generally, since $\omega^{n+k2^{r+1}} = (-1)^k \omega^n$ for any n we may write

$$c_{r,n+k2^{r+1}} = \sum_{\alpha} C_{r,\alpha,n} (-|\alpha|^{2^{r+1}})^k \tag{3.5}$$

for real constants $C_{r,\alpha,n} = |\alpha|^n (\omega^{n-2} + \overline{\omega}^{n-2})$ which do not depend on k, where the sum is over all 2^{r-1} values of $\alpha = 2/(1-\zeta)$ with positive imaginary part. This shows that for each r, each lacunary subsequence $\{c_{r,n+k2^{r+1}}\}_{k=1}^{\infty}$ satisfies the *same* linear recurrence of order 2^{r-1} with different initial conditions depending on n.

Consider now the $r = 2$ case of (2.5),

$$\sum_{n=0}^{\infty} c_{2,n} t^n = \frac{8t}{(1-2t)^4 + 1} = \frac{4t}{1 - 4t + 12t^2 - 16t^3 + 8t^4}. \tag{3.6}$$

We can use the recurrence $c_{2,n} = 4c_{2,n-1} - 12c_{2,n-2} + 16c_{2,n-3} - 8c_{2,n-4}$ with initial conditions $c_{2,1} = 4$, $c_{2,n} = 0$ for $n \leq 0$ to compute the first sixteen terms

$$\begin{array}{cccccccc} 4, & 16, & 16, & -64, & -224, & 0, & 1536, & 3072, \\ -4352, & -29696, & -29696, & 143360, & 489472, & 0, & -3342336, & -6684672, \end{array} \tag{3.7}$$

whose corresponding 2-ordinals are

$$\begin{array}{cccccccc} 2, & 4, & 4, & 6, & 5, & \infty, & 9, & 10, \\ 8, & 10, & 10, & 12, & 11, & \infty, & 16, & 17. \end{array} \tag{3.8}$$

By (3.4) we also have

$$c_{2,n} = |\alpha_1|^n (\omega_1^{n-2} + \overline{\omega}_1{}^{n-2}) + |\alpha_3|^n (\omega_3^{n-2} + \overline{\omega}_3{}^{n-2}) \tag{3.9}$$

where $\zeta^j = e^{ji\pi/4}$, $\alpha_j = 2/(1-\zeta^j)$, and $\omega_j = \alpha_j/|\alpha_j|$. When $n = 8k - 2$ we have $\omega_j^{n-2} = \pm i$ and thus $c_{2,n} = 0$. We compute directly that $|\alpha_1| = \sqrt{4 + 2\sqrt{2}}$ and $|\alpha_3| = \sqrt{4 - 2\sqrt{2}}$, so that $-|\alpha_1|^8 = -1088 - 768\sqrt{2}$ and $-|\alpha_3|^8 = -1088 + 768\sqrt{2}$. It follows from (3.5) that each lacunary subsequence $\{a_k\} = \{c_{2,n+8k}\}$ satisfies the recurrence $a_k = -2176a_{k-1} - 4096a_{k-2}$. From this we see that $c_{2,8k+2} = c_{2,8k+3}$ and $c_{2,8k} = 2c_{2,8k-1}$ for all k, because these relations hold for $k = 0, 1$ and the sequences $\{c_{2,n+8k}\}$ all satisfy the same second order recurrence. The recurrence $a_k = -2176a_{k-1} -$

$4096a_{k-2}$ for $\{a_k\} = \{c_{2,n+8k}\}$ also implies that $\mathrm{ord} c_{2,n+16} \geq \min\{7 + \mathrm{ord} c_{2,n+8}, 12 + \mathrm{ord} c_{2,n}\}$, with equality if $7+\mathrm{ord} c_{2,n+8}$ and $12+\mathrm{ord} c_{2,n}$ are different; by induction from (3.8) this shows that $\mathrm{ord} c_{2,n+8} = \mathrm{ord} c_{2,n} + 6$ when $n \not\equiv 0, -1, -2 \pmod 8$, which proves $\mathrm{ord} c_{2,n} = e_n$ in those cases. The sequence $\{c_{2,8k}\}$ is given by (3.9) in the form

$$c_{2,8k} = (-1)^k \sqrt{2} \left[(4 - 2\sqrt{2})^{4k} - (4 + 2\sqrt{2})^{4k} \right] \tag{3.10}$$

which we rewrite as

$$c_{2,8k} = (-1)^k \sqrt{2} (4 - 2\sqrt{2})^{4k} \left[1 - (3 + 2\sqrt{2})^{4k} \right]. \tag{3.11}$$

Since $\mathrm{ord}(4 - 2\sqrt{2}) = \frac{3}{2}$ we have $\mathrm{ord}(4 - 2\sqrt{2})^{4k} = 6k$. Since $(3 + 2\sqrt{2})^2 = 17 + 12\sqrt{2} = 1 + y$ with $\mathrm{ord} y = \frac{5}{2}$ we have $(3 + 2\sqrt{2})^{4k} = (1 + y)^{2k} \equiv 1 + 2ky \pmod{4ky\mathfrak{O}_2}$ and therefore $\mathrm{ord}(1 - (3 + 2\sqrt{2})^{4k}) = \frac{7}{2} + \mathrm{ord} k$. This shows that $\mathrm{ord} c_{2,8k} = 6k + \mathrm{ord} k + 4$, and therefore $\mathrm{ord} c_{2,8k-1} = 6k + \mathrm{ord} k + 3$. Thus we have determined $\mathrm{ord} c_{2,n}$ exactly, and we have $\mathrm{ord} c_{2,n} = e_n$ for all $n \not\equiv 6 \pmod 8$, whereas $c_{2,8k-2} = 0$ for all k.

The first sixteen values of $c_{3,n}$ are

$$\begin{array}{cccc} 8, & 64, & 64, & -1280, \\ -3968, & 25600, & 154624, & -364544, \\ -4791296, & 0, & 125894656, & 251789312, \\ -2804219904, & -12224036864, & 49230381056, & 419636969472, \end{array} \tag{3.12}$$

and their 2-ordinals are

$$\begin{array}{cccccccc} 3, & 6, & 6, & 8, & 7, & 10, & 10, & 12, \\ 10, & \infty, & 16, & 17, & 16, & 18, & 18, & 20. \end{array} \tag{3.13}$$

It follows that $\mathrm{ord} c_{2,n} = e_n < \mathrm{ord} c_{3,n}$ for $n \leq 16$ and $n \not\equiv 6 \pmod 8$; we may also use the lemma to verifty that $\mathrm{ord} c_{2,n} = e_n < 3 + \frac{7}{8}(n-1) \leq \mathrm{ord} c_{3,n}$ for $n > 16$ and $n \not\equiv 6 \pmod 8$. The lemma also shows that $\mathrm{ord} c_{2,n} = e_n < \mathrm{ord} c_{r,n}$ for all $n \not\equiv 6 \pmod 8$ and all $r \geq 4$. This proves Assertions (2) and (3), and therefore Theorem 2, in all cases except for $n = 8k - 2$.

For $j = 1, 3, 5, 7$ we now set $\zeta^j = e^{ji\pi/8}$, $\alpha_j = 2/(1 - \zeta^j)$, and $\omega_j = \alpha_j / |\alpha_j|$. We now consider the sum

$$c_{3,8k-2} = -\mathrm{Tr}_3(\zeta \alpha^{8k-2}) = \sum_{j=1,3,5,7} |\alpha_j|^{8k-2} (\omega_j^{8k-4} + \overline{\omega}_j^{8k-4}) \tag{3.14}$$

for arbitrary integers k; by (2.7) this agrees with (2.5) for $k > 0$. Since $\omega_j^{-2} = -\zeta^j$ we have $(\omega_j^{8k-4} + \overline{\omega}_j{}^{8k-4}) = \pm\sqrt{2}$ and we may compute

$$\begin{aligned} c_{3,8k-2} &= (\omega_1^{8k-4} + \overline{\omega}_1^{8k-4})\left[|\alpha_1|^{8k-2} - |\alpha_3|^{8k-2} - |\alpha_5|^{8k-2} + |\alpha_7|^{8k-2}\right] \\ &= (\omega_j^{8k-4} + \overline{\omega}_j{}^{8k-4})|\alpha_1|^{8k-2}\left[1 - \left|\frac{\alpha_3}{\alpha_1}\right|^{8k-2} - \left|\frac{\alpha_5}{\alpha_1}\right|^{8k-2} + \left|\frac{\alpha_7}{\alpha_1}\right|^{8k-2}\right] \\ &= (\omega_j^{8k-4} + \overline{\omega}_j{}^{8k-4})|\alpha_1|^{8k-2}\cdot f(k). \end{aligned} \tag{3.15}$$

We claim that the function $f(k)$ in brackets in (3.15) is an analytic function of k on $\mathbb{Z}_2$. We calculate

$$\left|\frac{\alpha_j}{\alpha_1}\right|^2 = \zeta^{j-1}\left(\frac{1-\zeta}{1-\zeta^j}\right)^2 = 1 - \frac{(1-\zeta^{j-1})(1-\zeta^{j+1})}{(1-\zeta^j)^2}. \tag{3.16}$$

Since a primitive 2^{r+1}-th root of unity ζ has $\mathrm{ord}(1-\zeta) = 2^{-r}$, (3.16) shows that for $j = 3, 5, 7$ we have $|\alpha_j/\alpha_1|^2 = 1 + y_j$ with $\mathrm{ord} y_3 = 1/2$, $\mathrm{ord} y_5 = 1/2$, and $\mathrm{ord} y_7 = 1$. This implies that $(1+y_j)^4 = 1 + Y_j$ with $\mathrm{ord} Y_j \geq 2$ for $j = 3, 5, 7$. Writing $\log(1+Y_j) = U_j$ with $\mathrm{ord} U_j \geq 2$ we have

$$(1+Y_j)^k = \exp(kU_j) = \sum_{i=0}^{\infty} \frac{U_j^i}{i!} k^i \tag{3.17}$$

as 2-adically analytic functions in k on $\{k \in K_3 : \mathrm{ord} k > -1\}$, hence analytic on $\mathbb{Z}_2$. Since

$$f(k) = 1 - \left|\frac{\alpha_1}{\alpha_3}\right|^2 (1+Y_3)^k - \left|\frac{\alpha_1}{\alpha_5}\right|^2 (1+Y_5)^k + \left|\frac{\alpha_1}{\alpha_7}\right|^2 (1+Y_7)^k \tag{3.18}$$

is a sum of 2-adic unit multiples of analytic functions on $\mathbb{Z}_2$, it is analytic on $\mathbb{Z}_2$, so we may write

$$f(k) = \sum_{i=0}^{\infty} a_i k^i \qquad \text{with} \qquad a_i = \delta_{i,0} - \left|\frac{\alpha_1}{\alpha_3}\right|^2 \frac{U_3^i}{i!} - \left|\frac{\alpha_1}{\alpha_5}\right|^2 \frac{U_5^i}{i!} + \left|\frac{\alpha_1}{\alpha_7}\right|^2 \frac{U_7^i}{i!} \in \mathfrak{O}_3 \tag{3.19}$$

for all $k \in \mathbb{Z}_2$. Since each $\mathrm{ord} U_j \geq 2$, we have $\mathrm{ord} a_i \geq 5$ for $i \geq 3$. We now consider $f(0)$, $f(1)$, and $f(2)$.

From (2.7), (3.14) we have

$$\begin{aligned} c_{3,-2} &= -\sum_\zeta \zeta\alpha^{-2} = -\frac{1}{4}\sum_\zeta \zeta(1-\zeta)^2 \\ &= -\frac{1}{4}\sum_\zeta \zeta + \frac{1}{2}\sum_\zeta \zeta^2 - \frac{1}{4}\sum_\zeta \zeta^3 = 0 + 0 + 0 = 0 \end{aligned} \tag{3.20}$$

and hence from (3.15) we get $f(0) = a_0 = 0$. Since $\mathrm{ord}(\sqrt{2}|\alpha_1|^{8k-2}) = \frac{1}{2} + \frac{7}{8}(8k-2) = 7k - \frac{5}{4}$ and $\mathrm{ord} c_{3,6} = \mathrm{ord}(25600) = 10$, we see from (3.15) that $\mathrm{ord} f(1) = 17/4$; since $\mathrm{ord} c_{3,14} = 18$ we have $\mathrm{ord} f(2) = 21/4$. Now from (3.19) we have

$$f(1) \equiv a_1 + a_2 \pmod{2^5\mathfrak{O}_3}; \qquad f(2) \equiv 2a_1 + 4a_2 \pmod{2^8\mathfrak{O}_3}. \tag{3.21}$$

If we had $\mathrm{ord}a_1 = \mathrm{ord}a_2 < 17/4$ then $\mathrm{ord}f(1) = 17/4$ would be possible in (3.21) but $\mathrm{ord}f(2) = 21/4$ would not; thus exactly one of $\mathrm{ord}a_1$, $\mathrm{ord}a_2$ is equal to $17/4$ and the other is larger than $17/4$. Supposing $\mathrm{ord}a_2 = 17/4$, $\mathrm{ord}a_1 > 17/4$ would make $\mathrm{ord}f(2) = 21/4$ false, so we must have $\mathrm{ord}a_1 = 17/4$ and $\mathrm{ord}a_2 > 17/4$. Then since $\mathrm{ord}a_i \geq 5$ for $i \geq 3$ by (3.19), we must have $\mathrm{ord}f(k) = \mathrm{ord}a_1 k = \mathrm{ord}k + 17/4$ for all $k \in \mathbb{Z}_2$. Combined with (3.15), this shows that $\mathrm{ord}c_{3,8k-2} = 7k + \mathrm{ord}k + 3$ for all $k \in \mathbb{Z}$, completing the proof of Assertion (2).

We calculate directly that $\mathrm{ord}c_{4,8k-2} = 12, 20, 29, 36, 42, 50, 60, 67$ for $1 \leq k \leq 8$, and then that $\mathrm{ord}c_{4,8k-2} > e_{8k-2}$ for $k > 8$ by the lemma; thus $\mathrm{ord}c_{4,8k-2} > e_{8k-2}$ for all positive integers k. We then calculate directly that $\mathrm{ord}c_{5,8k-2} = 14, 22$ for $1 \leq k \leq 2$, and $\mathrm{ord}c_{5,8k-2} > e_{8k-2}$ for $k > 2$ by the lemma; thus $\mathrm{ord}c_{5,8k-2} > e_{8k-2}$ for all positive integers k. Finally, the lemma shows that $\mathrm{ord}c_{r,8k-2} > e_{8k-2}$ for all positive integers k and all $r \geq 6$. This completes the proof of Assertion (3), and thus completes the proof of Theorem 2.

4. 2-adic analysis of Nörlund numbers

An analogue of Theorem 1 for the Nörlund numbers is as follows:

Theorem 3. *For each $r \geq 0$ let ζ_r denote any primitive 2^{r+1}-th root of unity. Then for all nonnegative integers n we have*

$$(-2)^n \frac{B_n^{(n)}}{n!} = -\sum_{r=0}^{\infty} \mathrm{Tr}_r \left(\left(\frac{2}{1-\zeta_r} \right)^n \right)$$

as a 2-adically convergent sum of integers.

The following result, which contains the conjectures of Adelberg [1] concerning the Nörlund numbers, may be derived from Theorem 3 by a method similar to that of §3:

Theorem 4. *For all nonnegative integers n, we have*

$$(-2)^n \frac{B_n^{(n)}}{n!} \equiv -1 + L_n 2^{[n/2]+1} + 2^{E_n} \qquad (\mathrm{mod}\ 2^{E_n+1}\mathbb{Z}_2)$$

where L_n is given by

$$L_n = \begin{cases} 1, & \text{if } n \equiv 3,4,5 \ (\text{mod } 8), \\ -1, & \text{if } n \equiv 0,1,7 \ (\text{mod } 8), \\ 0, & \text{if } n \equiv 2,6 \ (\text{mod } 8), \end{cases}$$

$E_3 = \infty$, and for $n \neq 3$, E_n is given by

n	$8k-2$	$8k-1$	$8k$	$8k+1$	$8k+2$	$8k+3$	$8k+4$	$8k+5$
E_n	$6k$	$6k+1$	$6k+2$	$6k+2$	$6k+3$	$6k+5$	$7k+6$	$6k+5$.

The values of L_n were called the *conjectured stable values* by Adelberg ([1], eq. (19)), who also observed the values of E_n via numerical computation ([1], eq. (20)).

We give a brief outline of the proofs of these two theorems. If we divide both sides of (2.3) by $1-2t$, we get

$$\frac{-2t}{(1-2t)\log(1-2t)} - \frac{1}{(1-t)(1-2t)} = \sum_{r=1}^{\infty} \frac{2^{r+1}t}{(1-2t)((1-2t)^{2^r}+1)} \tag{4.1}$$

as an identity in $\mathbb{Z}_2[[t]]$. In the partial fraction decomposition of the rational functions in (4.1), the terms with denominator $1-2t$ all cancel (the numerator is $2+\sum_{r=1}^{\infty} 2^r$, which is zero in $\mathbb{Z}_2$), since the singularity at $t=1/2$ in the left-most term in (4.5) is removable. Thus we are left with

$$\frac{-2t}{(1-2t)\log(1-2t)} + \frac{1}{1-t} = \sum_{r=1}^{\infty} \frac{-2^r((1-2t)^{2^r-1}+1)}{(1-2t)^{2^r}+1}, \tag{4.2}$$

which we rewrite as

$$\sum_{n=0}^{\infty} C_n t^n = \sum_{r=1}^{\infty}\sum_{n=0}^{\infty} C_{r,n} t^n \tag{4.3}$$

where $C_n = (-2)^n B_n^{(n)}/n! + 1$ as in [1] and, by partial fraction decomposition,

$$C_{r,n} = -\sum_{\zeta} \left(\frac{2}{1-\zeta}\right)^n = -\mathrm{Tr}_r\left(\left(\frac{2}{1-\zeta_r}\right)^n\right), \tag{4.4}$$

where the sum is over all primitive 2^{r+1}-th roots of unity $\zeta = \zeta_r$. Thus Theorem 1 implies Theorem 3. In fact the two are equivalent: Assuming Theorem 3, by (1.10) we have

$$\begin{aligned}(-2)^n b_n &= (-2)^n \frac{B_n^{(n)}}{n!} - 2(-2)^{n-1}\frac{B_{n-1}^{(n-1)}}{(n-1)!} \\ &= -\sum_{r=0}^{\infty} \mathrm{Tr}_r\left(\left(\frac{2}{1-\zeta_r}\right)^n - 2\left(\frac{2}{1-\zeta_r}\right)^{n-1}\right) \\ &= -\sum_{r=0}^{\infty} \mathrm{Tr}_r\left(\zeta_r\left(\frac{2}{1-\zeta_r}\right)^n\right),\end{aligned} \tag{4.5}$$

proving Theorem 1. Alternately, Theorem 3 may also be derived from Theorem 1 via (1.10).

To prove Theorem 4, we observe that the sequence $\{C_{r,n}\}_{n=0}^{\infty}$ satisfies the same linear recurrence as $\{c_{r,n}\}_{n=0}^{\infty}$, and the lemma of §2 remains valid with $c_{r,n}$ replaced by $C_{r,n}$. Theorem 4 then follows from the following three assertions:

(1). $C_{1,n} = L_n 2^{[n/2]+1}$ for all n;

(2). ord $(C_{2,n} + C_{3,n}) = E_n$ for all $n \neq 3$;

(3). ord $C_{r,n} > E_n$ for all $r \geq 4$ and all $n \neq 3$.

The value $E_3 = \infty$ may be determined by direct calculation (cf. [1], p. 55). The analogue of (3.2) for the C_n is

$$C_{1,n} = -((1-i)^n + (1+i)^n) = \begin{cases} -2(-4)^k, & \text{if } n = 4k, \\ -2(-4)^k, & \text{if } n = 4k+1, \\ 0, & \text{if } n = 4k+2, \\ -(-4)^{k+1}, & \text{if } n = 4k+3, \end{cases} \tag{4.6}$$

which proves Assertion (1), and the analogue of (3.4) is

$$C_{r,n} = -\mathrm{Tr}_r(\alpha^n) = -\sum_{\zeta} \alpha^n = -\sum_{\alpha} |\alpha|^n (\omega^n + \varpi^n) \tag{4.7}$$

where $\alpha = 2/(1-\zeta)$, $\omega = \alpha/|\alpha|$, the first sum is over all primitive 2^{r+1}-th roots of unity $\zeta = \zeta_r$, and the second sum is over all such values of α with positive imaginary part. This shows that $C_{r,n} = 0$ when $n \equiv 2^r \pmod{2^{r+1}}$, so in particular $C_{2,8k+4} = 0$ for all k. The recurrence $a_k = -2176 a_{k-1} - 4096 a_{k-2}$ for the lacunary subsequences $\{a_k\} = \{C_{2,n+8k}\}$ easily shows that $\mathrm{ord} C_{2,n} = E_n$ for all $n \not\equiv 4 \pmod 8$. The demonstration that $\mathrm{ord} C_{3,8k+4} = E_{8k+4}$ is a bit easier than the corresponding result for $c_{3,8k-2}$, because the analytic function $F(k)$ one obtains in the calculation analogous to (3.15) does not vanish at zero; this is why the formulas for E_{8k+j} do not depend on ord$k$, as do those for the $j = 0, -1, -2$ cases of e_{8k+j}.

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