A 2-adic formula for Bernoulli numbers of the second kind
and for the Nörlund numbers

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Abstract

We give a formula expressing Bernoulli numbers of the second kind as 2-adically convergent sums of traces of algebraic integers. We use this formula to prove and explain the formulas and conjectures of Adelberg concerning the initial 2-adic digits of these numbers. We also give analogous results for the Nörlund numbers.

Keywords: Bernoulli numbers of second kind, Nörlund numbers, Cauchy numbers, congruences, p-adic analysis

1. Introduction

The Bernoulli numbers of the second kind \( b_n \) are the rational numbers determined ([2], [5]) by the generating function

\[
\frac{t}{\log(1 + t)} = \sum_{n=0}^{\infty} b_n t^n.
\]  

(1.1)

The numbers \( n!b_n \) have also been called Cauchy numbers of the first type ([3], [7]), and may be defined by

\[
n!b_n = \int_0^1 x(x - 1)(x - 2) \cdots (x - n + 1) \, dx.
\]  

(1.2)

The first few values are \( b_0 = 1, b_1 = 1/2, b_2 = -1/12, b_3 = 1/24, b_4 = -19/720, b_5 = 3/160. \) Their applications in number theory include

\[
\sum_{n=1}^{\infty} (-1)^{n+1} \frac{b_n}{n} = \gamma;
\]

\[
1 + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{b_n H_n}{n} = \frac{\pi^2}{6},
\]  

(1.3)

(cf. [7], §4) where \( H_n = \sum_{k=1}^{n} 1/k \) is the \( n \)-th harmonic number and \( \gamma = \lim_{n \to \infty} (H_n - \log n) \) denotes Euler’s constant. Among the interesting arithmetic properties of these numbers are the large initial gaps in their 2-adic expansions; for example,

\[
2^{15}b_{15} \equiv -1 + 2^8 + 2^{16} \pmod{2^{17} \mathbb{Z}_2},
\]  

(1.4)

\[
2^{20}b_{20} \equiv 1 + 2^{18} \pmod{2^{19} \mathbb{Z}_2},
\]  

(1.5)

\[
2^{30}b_{30} \equiv 1 - 2^{16} + 2^{33} \pmod{2^{34} \mathbb{Z}_2}.
\]  

(1.6)
This property was studied recently by Adelberg [1], who proved that \((-2)^n b_n \equiv 1 \pmod{2^{[n/2] + 1} \mathbb{Z}_2}\) where \([x]\) is the greatest integer function, and showed that this congruence is best possible if 4 does not divide \(n\); this essentially determines the first gap. He also made several conjectures concerning the second gap on the basis of numerical computation. In this note we give a simple 2-adic formula for the numbers \(b_n\) and use it to verify those conjectures. In fact we prove the following:

**Theorem 2.** For all nonnegative integers \(n\), we have

\[
(-2)^n b_n \equiv 1 + \varepsilon_n 2^{[n/2] + 1} + 2^{e_n} \pmod{2^{e_n + 1} \mathbb{Z}_2}
\]

where \(\varepsilon_n\) is given by

\[
\varepsilon_n = \begin{cases} 
1, & \text{if } n \equiv 1, 2, 3 \pmod{8}, \\
-1, & \text{if } n \equiv 5, 6, 7 \pmod{8}, \\
0, & \text{if } n \equiv 0, 4 \pmod{8},
\end{cases}
\]

and \(e_n\) is given by

<table>
<thead>
<tr>
<th>(n)</th>
<th>(8k - 2)</th>
<th>(8k - 1)</th>
<th>(8k)</th>
<th>(8k + 1)</th>
<th>(8k + 2)</th>
<th>(8k + 3)</th>
<th>(8k + 4)</th>
<th>(8k + 5)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(e_n)</td>
<td>7k + \text{ord}8k</td>
<td>6k + \text{ord}8k</td>
<td>6k + \text{ord}16k</td>
<td>6k + 2</td>
<td>6k + 4</td>
<td>6k + 4</td>
<td>6k + 6</td>
<td>6k + 5,</td>
</tr>
</tbody>
</table>

where \(\text{ord}\) is the 2-adic valuation normalized by \(\text{ord}2 = 1\).

This theorem gives a simple formula which precisely predicts at least the first \(\frac{3n}{4}\) 2-adic digits of any \(b_n\). The values of \(\varepsilon_n\) were called the conjectured stable congruences for the \(b_n\) by Adelberg ([1], p. 57), who also observed the above values of \(e_n\) through numerical computation, although he declined to state a formula for \(e_n\) in the cases \(n \equiv 0, -1, -2 \pmod{8}\) where the formula involves \(\text{ord}k\). Although the calculation of \(e_n\) in these three cases requires slightly more delicate analysis than the others, these congruences all follow by essentially the same line of argument from the following formula:

**Theorem 1.** For each \(r \geq 0\) let \(\zeta_r\) denote any primitive \(2^{r+1}\)-th root of unity. Then for all nonnegative integers \(n\) we have

\[
(-2)^n b_n = -\sum_{r=0}^{\infty} \text{Tr}_r \left( \zeta_r \left( \frac{2}{1 - \zeta_r} \right)^n \right)
\]

as a 2-adically convergent sum of integers, where \(\text{Tr}_r\) denotes the trace map from \(\mathbb{Q}(\zeta_r)\) to \(\mathbb{Q}\).

The Bernoulli numbers of order \(w\), \(B_n^{(w)}\), are the rational numbers defined [8] by the generating function

\[
\left( \frac{t}{e^t - 1} \right)^w = \sum_{n=0}^{\infty} B_n^{(w)} \frac{t^n}{n!}.
\]  

(1.7)
For $n = w$ the numbers $B_n^{(n)}$ are called Nörlund numbers [4], or Cauchy numbers of the second type ([3], [7]), and may be determined by the generating function

$$
\frac{t}{(1 + t) \log(1 + t)} = \sum_{n=0}^{\infty} B_n^{(n)} \frac{t^n}{n!}
$$

(1.8) (cf. [8]). The first few values are $B_0^{(0)} = 1$, $B_1^{(1)} = -1/2$, $B_2^{(2)} = 5/6$, $B_3^{(3)} = -9/4$, $B_4^{(4)} = 251/30$, $B_5^{(5)} = -475/12$. One important role they play in combinatorial analysis is through the formula

$$
B_n^{(n)} = \int_0^1 (x - 1) (x - 2) \cdots (x - n) \, dx
$$

(1.9) (cf. [8]). They are related to the Bernoulli numbers of the second kind via the formulas [4]

$$
\frac{B_n^{(n)}}{n!} = \sum_{j=0}^{n} (-1)^{n-j} b_j \quad \text{and} \quad b_n = \frac{B_n^{(n)}}{n!} + \frac{B_n^{(n-1)}}{(n-1)!}.
$$

(1.10)

Adelberg also studied their 2-adic expansion in [1] and proved that $(-2)^n B_n^{(n)} / n! \equiv -1$ (mod $2^{n/2} + 1 \mathbb{Z}_2$), with this congruence being best possible if $n \not\equiv 2$ (mod 4) ([1], Theorem 2 and Corollary 1 of §3); he further conjectured “stable congruences” for them which were supported by numerical calculations. In §4 we give a formula similar to Theorem 1 for $B_n^{(n)} / n!$ which furnishes proof of Adelberg’s conjectures concerning the 2-adic digits of these numbers.

2. Proof of 2-adic formula

Throughout this paper $\mathbb{Z}_2$ will denote the ring of 2-adic integers and $\mathbb{Q}_2$ will denote the field of 2-adic numbers. Clearly $\zeta = \zeta_r$ is a primitive $2^{r+1}$-th root of unity if and only if $\zeta^{2^r} = 1$, so the minimal polynomial for $\zeta_r$ over $\mathbb{Q}$ is the $2^{r+1}$-th cyclotomic polynomial $\Phi_{2^{r+1}}(t) = t^{2^r} + 1$. It is well known that $\mathbb{Q}(\zeta_r)$ is an abelian extension of $\mathbb{Q}$ with Galois group $\text{Gal}(\mathbb{Q}(\zeta_r)/\mathbb{Q}) = \{\sigma_j : j \in (\mathbb{Z}/2^{r+1}\mathbb{Z})^\times\} \cong (\mathbb{Z}/2^{r+1}\mathbb{Z})^\times$, where $\sigma_j$ is the automorphism of $\mathbb{Q}(\zeta_r)$ induced by $\zeta_r \mapsto \zeta_r^j$.

Since $1 - \zeta$ is a root of the 2-Eisenstein polynomial $\Phi_{2^{r+1}}(1 - t) = 2 - 2^r t + \cdots - 2^r t^{2^{r-1}} + t^{2^r}$ we have $\text{ord}(1 - \zeta) = 1/2^r$; hence the degree-$2^r$ extension $K_r = \mathbb{Q}_2(\zeta_r)$ of $\mathbb{Q}_2$ is totally ramified, with ring of integers $\mathcal{O}_r = \{x \in K_r : \text{ord} x \geq 0\}$, maximal ideal $\mathfrak{p}_r = \{x \in K_r : \text{ord} x \geq 1/2^r\}$, and residue class field $\mathcal{O}_r/\mathfrak{p}_r$ isomorphic to $\mathbb{Z}/2\mathbb{Z}$ (cf. [6]). It follows that for $x, y \in K_r$ we have $\text{ord}(x + y) \geq \min\{\text{ord} x, \text{ord} y\}$ with equality if and only if $\text{ord} x \neq \text{ord} y$.  


Proof of Theorem 1. For any $r \geq 0$, 
\[
\frac{-2^{r+2} t}{(1 - 2t)^{2^{r+1}} - 1} - \frac{-2^{r+1} t}{(1 - 2t)^{2^r} - 1} = \frac{2^{r+1} t}{(1 - 2t)^{2^r} + 1}.
\]  
(2.1)
If we sum this equation from $r = 1$ to $r = s$, the left side telescopes, yielding 
\[
\frac{-2^{s+2} t}{(1 - 2t)^{2^{s+1}} - 1} - \frac{1}{1 - t} = \sum_{r=1}^{s} \frac{2^{r+1} t}{(1 - 2t)^{2^r} + 1}.
\]  
(2.2)
Since $((1 - 2t)^2 + 1)/2$ is a unit in the power series ring $\mathbb{Z}_2[[t]]$, the $r$-th term in the sum in (2.2) lies in $2^r \mathbb{Z}_2[[t]]$, hence the 2-adic limit of partial sums as $s \to \infty$ exists. Since $((1+t)^n - 1) / a \to \log(1+t)$ as $a \to 0$ 2-adically, we have 
\[
\frac{-2t}{\log(1 - 2t)} - \frac{1}{1 - t} = \sum_{r=1}^{\infty} \frac{2^{r+1} t}{(1 - 2t)^{2^r} + 1}
\]  
(2.3)
as an identity in $\mathbb{Z}_2[[t]]$. Expanding the left side of (2.3) as a power series gives 
\[
\sum_{n=0}^{\infty} c_n t^n = \sum_{r=1}^{\infty} \frac{2^{r+1} t}{(1 - 2t)^{2^r} + 1}
\]  
(2.4)
where $c_n = (-2)^n b_n - 1$ for all $n$, as in [1]. We define rational integers $c_{r,n}$ by 
\[
\sum_{n=0}^{\infty} c_{r,n} t^n = \frac{2^{r+1} t}{(1 - 2t)^{2^r} + 1}
\]  
(2.5)
so that $c_n = \sum_{r=1}^{\infty} c_{r,n}$ as a convergent sum in $\mathbb{Z}_2$ for all $n \geq 0$.

If $Q(t) = \prod_{i=1}^{n} (1 - \alpha_i t)$ is a polynomial of degree $n$ with distinct roots and $P(t)$ is a polynomial of degree less than $n$, it is easily seen that there is a partial fraction decomposition of $P(t)/Q(t)$ as $\sum_{i=1}^{n} a_i/(1 - \alpha_i t)$ where $a_i = -\alpha_i P(\alpha_i^{-1})/Q'(\alpha_i^{-1})$ for all $i$. Thus since $(1 - 2t)^{2^r} + 1 = 0$ whenever $2t = 1 - \zeta$, for a primitive $2^{r+1}$-th root of unity $\zeta_r$, we have for each $r$ by partial fraction decomposition 
\[
\sum_{n=0}^{\infty} c_{r,n} t^n = \frac{2^{r+1} t}{(1 - 2t)^{2^r} + 1} = \sum_{\zeta} \frac{-\zeta}{1 - \alpha \zeta} = \sum_{n=0}^{\infty} \sum_{\zeta} -\zeta \alpha^n t^n
\]  
(2.6)
where $\alpha = 2/(1 - \zeta)$ and the sums indexed by $\zeta$ are taken over all primitive $2^{r+1}$-th roots of unity $\zeta = \zeta_r$. Therefore 
\[
c_{r,n} = -\sum_{\zeta} \zeta \left( \frac{2}{1 - \zeta} \right)^n = -\text{Tr}_{r} \left( \zeta_r \left( \frac{2}{1 - \zeta_r} \right)^n \right)
\]  
(2.7)
for all $r, n$, where each $\zeta_r$ denotes any fixed primitive $2^{r+1}$-th root of unity. This completes the proof.

**Remarks.** One may use this method to derive similar $p$-adic formulas for $b_n$ for any prime $p$. Heuristically the proof of this formula gives

$$(-2)^n b_n = 1 + \sum_{r=1}^{\infty} c_{r,n} = -\sum \zeta \left( \frac{2}{1 - \zeta} \right)^n$$

(2.8)

where the latter sum is over all nontrivial roots of unity $\zeta$ of 2-power order; however in this form the latter sum is 2-adically divergent, since the terms are bounded away from zero. By grouping terms together with their Galois conjugates, the trace maps combine the terms in (2.8) to produce the 2-adically convergent sum of the theorem. A consequence of this theorem is that for each $n$ the sequence of traces $\{\text{Tr}_r(\zeta_r(2/(1 - \zeta_r))^n)\}$ converges 2-adically to zero as $r \to \infty$, which is a bit unexpected in light of the fact that $\{\zeta_r(2/(1 - \zeta_r))^n\}$ is 2-adically divergent as $r \to \infty$ for every $n$.

The dependence of $e_n$ and $e_n$ on the residue class of $n$ modulo 8 will be explained more or less by this expression of $c_n$ as a 2-adically convergent sum of the linearly recurrent sequences $\{c_{r,n}\}_{n=0}^{\infty}$. From (2.5) we see that for each $r$ the sequence $\{c_{r,n}\}$ satisfies a linear recurrence of order $2^r$, the reciprocal roots $\alpha$ of whose characteristic polynomial all have 2-adic ordinal $(2^r - 1)/2^r$.

**Lemma.** With $c_{r,n}$ as defined in (2.5) we have

$$\text{ord}_{r,n} \geq \left\lfloor r + (n - 1) \frac{2^r - 1}{2^r} \right\rfloor$$

for all positive integers $r$ and $n$, where $\lfloor x \rfloor$ denotes the least integer not less than $x$.

**Proof.** Write the characteristic polynomial $P_r(t) = ((1-2t)^{2^r} + 1)/2 = 1 - 2^rt + \cdots + 2^{2^r-1}t^{2^r} \in \mathbb{Z}[t]$.

If we introduce a change of variables $u = 2t/(1-\zeta_r)$ then $P_r(t) \in \mathcal{O}_r[u]$ with constant term 1, so that $P_r(t)$ is a unit in $\mathcal{O}_r[[u]]$. It follows that $P_r(t)^{-1} = \sum a_{r,n} t^n \in \mathcal{O}_r[[t]]$ with $\text{ord}_{a_{r,n}} \geq n(2^r - 1)/2^r$.

Since $c_{r,n} = 2^r a_{r,n-1} \in \mathbb{Z}$ the statement of the lemma follows.

**Remark.** Adelberg’s theorem ([1], Theorem 2) that $(-2)^n b_n \equiv 1 \pmod{2^{[n/2]+1}\mathbb{Z}_2}$ follows immediately from this lemma since $[n/2] + 1 = \lfloor 1 + (n - 1)/2 \rfloor$.

3. Proof of 2-adic congruences
In order to prove Theorem 2 we write

\[ c_n = c_{1,n} + c_{2,n} + c_{3,n} + \sum_{r=4}^{\infty} c_{r,n} \]  

where \( c_n = (-2)^n b_n - 1 \) and \( c_{r,n} \) is as defined in (2.5). Theorem 2 then follows from the following three assertions:

1. \( c_{1,n} = \varepsilon_n 2^{[n/2]+1} \) for all \( n \);
2. \( \text{ord} (c_{2,n} + c_{3,n}) = \varepsilon_n \) for all \( n \);
3. \( \text{ord} c_{r,n} > \varepsilon_n \) for all \( r \geq 4 \) and all \( n \).

A straightforward calculation from the \( r = 1 \) case of (2.5) or (2.7) yields

\[ c_{1,n} = i((1 - i)^n - (1 + i)^n) = \begin{cases} 
0, & \text{if } n = 4k, \\
2(-4)^k, & \text{if } n = 4k + 1, \\
-(4)^{k+1}, & \text{if } n = 4k + 2, \\
-(4)^{k+1}, & \text{if } n = 4k + 3, 
\end{cases} \]  

where \( i^2 = -1 \). Assertion (1) follows directly; from this calculation and the \( r \geq 2 \) cases of the lemma we see that Adelberg’s congruence \( (-2)^n b_n \equiv 1 \mod 2^{[n/2]+1} \mathbb{Z}_2 \) is indeed best possible in all cases except when \( 4 \mid n \) ([1], §3, Corollary 2). Indeed the lemma already proves that \( (-2)^n b_n \equiv 1 + \varepsilon_n 2^{[n/2]+1} \mod 2^{[3(n+5)/4]} \mathbb{Z}_2 \), which contains the “conjectured stable congruences” of Adelberg ([1], p. 57). The remainder of this section is devoted to the calculation of the exact modulus \( e_n \).

If \( \zeta = \zeta_r \) denotes a primitive \( 2^r+1 \)-th root of unity and \( \alpha = 2/(1 - \zeta) \), then \( -\alpha \zeta = \overline{\alpha} \) and therefore \( \alpha^2(-\zeta) = |\alpha|^2 \) (where \( \overline{\alpha} \) denotes complex conjugate and \( |\alpha| \) denotes complex absolute value). It follows that \( \omega = \alpha/|\alpha| \) is a primitive \( 2^{r+2} \)-th root of unity and in fact \( \omega^{-2} = -\zeta \). We may therefore rewrite (2.7) as

\[ c_{r,n} = -\text{Tr}_r(\zeta \alpha^n) = -\sum_{\zeta} \zeta \alpha^n = \sum_{\alpha} |\alpha|^n \omega^{n-2} \]  

where the latter sum is over all \( 2^r \) values of \( \alpha = 2/(1 - \zeta) \) for primitive \( 2^{r+1} \)-th roots of unity \( \zeta \), with \( \omega = \alpha/|\alpha| \). By pairing each such \( \alpha \) with its complex conjugate we may write

\[ c_{r,n} = \sum_{\alpha} |\alpha|^n (\omega^{n-2} + \overline{\omega}^{n-2}) \]  

where the sum is now over all \( 2^{r-1} \) such values of \( \alpha \) with positive imaginary part.
We mention two immediate consequences of this version of the formula. First, it directly implies that \( c_{r,n} = 0 \) when \( n \equiv 2^r + 2 \) (mod \( 2^{r+1} \)), because then each factor \( \omega^{n-2} + \overline{\omega}^{n-2} \) = 0. More generally, since \( \omega^{n+k2^{r+1}} = (-1)^k \omega^n \) for any \( n \) we may write

\[
c_{r,n+k2^{r+1}} = \sum_{\alpha} C_{r,\alpha,n} (-|\alpha|^{2^{r+1}})^k
\]

for real constants \( C_{r,\alpha,n} = |\alpha|^n (\omega^{n-2} + \overline{\omega}^{n-2}) \) which do not depend on \( k \), where the sum is over all \( 2^{r-1} \) values of \( \alpha = 2/(1-\zeta) \) with positive imaginary part. This shows that for each \( r \), each lacunary subsequence \( \{c_{r,n+k2^{r+1}}\}_{k=1}^{\infty} \) satisfies the same linear recurrence of order \( 2^{r-1} \) with different initial conditions depending on \( n \).

Consider now the \( r = 2 \) case of (2.5),

\[
\sum_{n=0}^{\infty} c_{2,n}t^n = \frac{8t}{(1-2t)^4 + 1} = \frac{4t}{1 - 4t + 12t^2 - 16t^3 + 8t^4}.
\]  

(3.6)

We can use the recurrence \( c_{2,n} = 4c_{2,n-1} - 12c_{2,n-2} + 16c_{2,n-3} - 8c_{2,n-4} \) with initial conditions \( c_{2,1} = 4 \), \( c_{2,n} = 0 \) for \( n \leq 0 \) to compute the first sixteen terms

\[
4, \quad 16, \quad 16, \quad -64, \quad -224, \quad 0, \quad 1536, \quad 3072, \quad -4352, \quad -29696, \quad -29696, \quad 143360, \quad 489472, \quad 0, \quad -3342336, \quad -6684672,
\]  

(3.7)

whose corresponding 2-ordinals are

\[
2, \quad 4, \quad 4, \quad 5, \quad 6, \quad 9, \quad 10, \quad 8, \quad 10, \quad 12, \quad 11, \quad \infty, \quad 16, \quad 17.
\]  

(3.8)

By (3.4) we also have

\[
c_{2,n} = |\alpha_1|^n (\omega_1^{n-2} + \overline{\omega_1}^{n-2}) + |\alpha_3|^n (\omega_3^{n-2} + \overline{\omega_3}^{n-2})
\]

(3.9)

where \( \zeta_j = e^{j\pi/4}, \alpha_j = 2/(1 - \zeta_j) \), and \( \omega_j = \alpha_j/|\alpha_j| \). When \( n = 8k - 2 \) we have \( \omega_j^{n-2} = \pm i \) and thus \( c_{2,n} = 0 \). We compute directly that \( |\alpha_1| = \sqrt{4 + 2\sqrt{2}} \) and \( |\alpha_3| = \sqrt{4 - 2\sqrt{2}} \), so that \( -|\alpha_1|^8 = -1088 - 768\sqrt{2} \) and \( -|\alpha_3|^8 = -1088 + 768\sqrt{2} \). It follows from (3.5) that each lacunary subsequence \( \{a_k\} = \{c_{2,n+8k}\} \) satisfies the recurrence \( a_k = -2176a_{k-1} - 4096a_{k-2} \). From this we see that \( c_{2,8k+2} = c_{2,8k+3} \) and \( c_{2,8k} = 2c_{2,8k-1} \) for all \( k \), because these relations hold for \( k = 0, 1 \) and the sequences \( \{c_{2,n+8k}\} \) all satisfy the same second order recurrence. The recurrence \( a_k = -2176a_{k-1} -
4096a_{k-2}$ for $\{a_k\} = \{c_{2,n+8k}\}$ also implies that $\text{ord}_{2,n+16} \geq \min\{7 + \text{ord}_{2,n+8}, 12 + \text{ord}_{2,n}\}$, with equality if $7 + \text{ord}_{2,n+8}$ and $12 + \text{ord}_{2,n}$ are different; by induction from (3.8) this shows that $\text{ord}_{2,n+8} = \text{ord}_{2,n} + 6$ when $n \not\equiv 0, -1, -2 \pmod{8}$, which proves $\text{ord}_{2,n} = e_n$ in those cases. The sequence $\{c_{2,8k}\}$ is given by (3.9) in the form

$$c_{2,8k} = (-1)^k \sqrt{2} \left[ (4 - 2 \sqrt{2})^{4k} - (4 + 2 \sqrt{2})^{4k} \right]$$

(3.10)

which we rewrite as

$$c_{2,8k} = (-1)^k \sqrt{2} (4 - 2 \sqrt{2})^{4k} \left[ 1 - (3 + 2 \sqrt{2})^{4k} \right].$$

(3.11)

Since $\text{ord}(4 - 2 \sqrt{2}) = \frac{3}{2}$ we have $\text{ord}(4 - 2 \sqrt{2})^{4k} = 6k$. Since $(3 + 2 \sqrt{2})^2 = 17 + 12 \sqrt{2} = 1 + y$ with $\text{ord} y = \frac{5}{2}$ we have $(3 + 2 \sqrt{2})^{4k} = (1 + y)^{2k} \equiv 1 + 2ky \pmod{4kyD_2}$ and therefore $\text{ord}(1 - (3 + 2 \sqrt{2})^{4k}) = \frac{7}{2} + \text{ord} k$. This shows that $\text{ord}_{2,8k} = 6k + \text{ord} k + 4$, and therefore $\text{ord}_{2,8k-1} = 6k + \text{ord} k + 3$. Thus we have determined $\text{ord}_{2,n}$ exactly, and we have $\text{ord}_{2,n} = e_n$ for all $n \not\equiv 6 \pmod{8}$, whereas $c_{2,8k-2} = 0$ for all $k$.

The first sixteen values of $c_{3,n}$ are

$$
\begin{align*}
8, & \quad 64, & \quad 64, & \quad -1280, \\
-3968, & \quad 25600, & \quad 154624, & \quad -364544, \\
-4791296, & \quad 0, & \quad 125894656, & \quad 251789312, \\
-2804219904, & \quad -12224036864, & \quad 49230381056, & \quad 419636969472,
\end{align*}
$$

(3.12)

and their 2-ordinals are

$$3, \quad 6, \quad 6, \quad 8, \quad 7, \quad 10, \quad 10, \quad 12, \quad 10, \quad \infty, \quad 16, \quad 17, \quad 16, \quad 18, \quad 18, \quad 20.$$

(3.13)

It follows that $\text{ord}_{2,n} = e_n < \text{ord}_{3,n}$ for $n \leq 16$ and $n \not\equiv 6 \pmod{8}$; we may also use the lemma to verify that $\text{ord}_{2,n} = e_n < 3 + \frac{7}{8}(n - 1) \leq \text{ord}_{3,n}$ for $n > 16$ and $n \not\equiv 6 \pmod{8}$. The lemma also shows that $\text{ord}_{2,n} = e_n < \text{ord}_{r,n}$ for all $n \not\equiv 6 \pmod{8}$ and all $r \geq 4$. This proves Assertions (2) and (3), and therefore Theorem 2, in all cases except for $n = 8k - 2$.

For $j = 1, 3, 5, 7$ we now set $\zeta^j = e^{2j\pi/8}$, $\alpha_j = 2/(1 - \zeta^j)$, and $\omega_j = \alpha_j/|\alpha_j|$. We now consider the sum

$$c_{3,8k-2} = -\text{Tr}_3(\zeta \alpha^{8k-2}) = \sum_{j=1,3,5,7} |\alpha_j|^{8k-2} (\omega_j^{8k-4} + \overline{\omega}_j^{8k-4})$$

(3.14)
for arbitrary integers \( k \); by (2.7) this agrees with (2.5) for \( k > 0 \). Since \( \omega_j^{-2} = -\zeta_j \) we have
\[
(\omega_j^{8k-4} + \omega_j^{8k-4}) = \pm \sqrt{2}
\]
and we may compute
\[
c_{3,8k-2} = (\omega_j^{8k-4} + \omega_j^{8k-4}) [\alpha_1^{8k-2} - |\alpha_3|^{8k-2} - |\alpha_5|^{8k-2} + |\alpha_7|^{8k-2}]
\]
\[
= (\omega_j^{8k-4} + \omega_j^{8k-4}) |\alpha_1|^{8k-2} \left[ 1 - \frac{|\alpha_3|}{|\alpha_1|}^{8k-2} - \frac{|\alpha_5|}{|\alpha_1|}^{8k-2} + \frac{|\alpha_7|}{|\alpha_1|}^{8k-2} \right] (3.15)
\]
\[
= (\omega_j^{8k-4} + \omega_j^{8k-4}) |\alpha_1|^{8k-2} \cdot f(k).
\]
We claim that the function \( f(k) \) in brackets in (3.15) is an analytic function of \( k \) on \( \mathbb{Z}_2 \). We calculate
\[
\frac{|\alpha_j|}{|\alpha_1|}^2 = \zeta^{-j} \left( \frac{1 - \zeta}{1 - \zeta_j} \right)^2 = 1 - \frac{(1 - \zeta_j^{-1})(1 - \zeta_j)}{(1 - \zeta_j)^2}. \quad (3.16)
\]
Since a primitive \( 2^{r+1} \)-th root of unity \( \zeta \) has ord\((1 - \zeta) = 2^{-r} \), (3.16) shows that for \( j = 3, 5, 7 \) we have \(|\alpha_j/\alpha_1|^2 = 1 + y_j \) with ord\( y_3 = 1/2 \), ord\( y_5 = 1/2 \), and ord\( y_7 = 1 \). This implies that \((1 + y_j)^k = 1 + Y_j \) with ord\( Y_j \geq 2 \) for \( j = 3, 5, 7 \). Writing \( \log(1 + Y_j) = U_j \) with ord\( U_j \geq 2 \) we have
\[
(1 + Y_j)^k = \exp(kU_j) = \sum_{i=0}^{\infty} \frac{U_j^i}{i!}k^i \quad (3.17)
\]
as \( 2 \)-adically analytic functions in \( k \) on \( \{ k \in K_3 : \text{ord}_2 k > -1 \} \), hence analytic on \( \mathbb{Z}_2 \). Since
\[
f(k) = 1 - \frac{|\alpha_1|}{|\alpha_3|}^2 (1 + Y_3)^k - \frac{|\alpha_1|}{|\alpha_5|}^2 (1 + Y_5)^k + \frac{|\alpha_1|}{|\alpha_7|}^2 (1 + Y_7)^k \quad (3.18)
\]
is a sum of \( 2 \)-adic unit multiples of analytic functions on \( \mathbb{Z}_2 \), it is analytic on \( \mathbb{Z}_2 \), so we may write
\[
f(k) = \sum_{i=0}^{\infty} a_ik^i \quad \text{with} \quad a_i = \delta_{i,0} - \frac{|\alpha_1|}{|\alpha_3|}^2 \frac{U_3^i}{i!} - \frac{|\alpha_1|}{|\alpha_5|}^2 \frac{U_5^i}{i!} + \frac{|\alpha_1|}{|\alpha_7|}^2 \frac{U_7^i}{i!} \in \mathcal{O}_3 \quad (3.19)
\]
for all \( k \in \mathbb{Z}_2 \). Since each ord\( U_j \geq 2 \), we have ord\( a_i \geq 5 \) for \( i \geq 3 \). We now consider \( f(0), f(1), \) and \( f(2) \).

From (2.7), (3.14) we have
\[
c_{3,-2} = - \sum_{\zeta} \zeta \alpha^{-2} = - \frac{1}{4} \sum_{\zeta} \zeta (1 - \zeta)^2
\]
\[
= - \frac{1}{4} \sum_{\zeta} \zeta^2 + \frac{1}{2} \sum_{\zeta} \zeta^2 - \frac{1}{4} \sum_{\zeta} \zeta^3 = 0 + 0 + 0 = 0 \quad (3.20)
\]
and hence from (3.15) we get \( f(0) = a_0 = 0 \). Since ord\((\sqrt{2}|\alpha_1|^{8k-2}) = \frac{1}{2} + \frac{7}{6}(8k - 2) = 7k - \frac{5}{3} \) and ord\( c_{3,6} = \text{ord}(25600) = 10 \), we see from (3.15) that ord\( f(1) = 17/4 \); since ord\( c_{3,14} = 18 \) we have ord\( f(2) = 21/4 \). Now from (3.19) we have
\[
f(1) \equiv a_1 + a_2 \pmod{2^5 \mathcal{O}_3}; \quad f(2) \equiv 2a_1 + 4a_2 \pmod{2^8 \mathcal{O}_3}. \quad (3.21)
\]
If we had \( \text{ord}_a = \text{ord}_b < 17/4 \) then \( \text{ord}_f(1) = 17/4 \) would be possible in (3.21) but \( \text{ord}_f(2) = 21/4 \) would not; thus exactly one of \( \text{ord}_a \), \( \text{ord}_b \) is equal to 17/4 and the other is larger than 17/4. Supposing \( \text{ord}_a = 17/4, \text{ord}_b > 17/4 \) would make \( \text{ord}_f(2) = 21/4 \) false, so we must have \( \text{ord}_a = 17/4 \) and \( \text{ord}_b > 17/4 \). Then since \( \text{ord}_a \geq 5 \) for \( i \geq 3 \) by (3.19), we must have \( \text{ord}_f(k) = \text{ord}_a k = \text{ord} k + 17/4 \) for all \( k \in \mathbb{Z}_2 \). Combined with (3.15), this shows that \( \text{ord}_c, 8k-2 = 7k + \text{ord} k + 3 \) for all \( k \in \mathbb{Z} \), completing the proof of Assertion (2).

We calculate directly that \( \text{ord}_c, 8k-2 = 12, 20, 29, 36, 42, 50, 60, 67 \) for \( 1 \leq k \leq 8 \), and then that \( \text{ord}_c, 8k-2 > e_{8k-2} \) for \( k > 8 \) by the lemma; thus \( \text{ord}_c, 8k-2 > e_{8k-2} \) for all positive integers \( k \). We then calculate directly that \( \text{ord}_c, 5, 8k-2 = 14, 22 \) for \( 1 \leq k \leq 2 \), and \( \text{ord}_c, 5, 8k-2 > e_{8k-2} \) for \( k > 2 \) by the lemma; thus \( \text{ord}_c, 5, 8k-2 > e_{8k-2} \) for all positive integers \( k \). Finally, the lemma shows that \( \text{ord}_c, r, 8k-2 > e_{8k-2} \) for all positive integers \( k \) and all \( r \geq 6 \). This completes the proof of Assertion (3), and thus completes the proof of Theorem 2.

4. 2-adic analysis of Nörlund numbers

An analogue of Theorem 1 for the Nörlund numbers is as follows:

**Theorem 3.** For each \( r \geq 0 \) let \( \zeta_r \) denote any primitive \( 2^{r+1} \)-th root of unity. Then for all nonnegative integers \( n \) we have

\[
(-2)^n \frac{B_n^{(n)}}{n!} = -\sum_{r=0}^{\infty} \text{Tr}_r \left( \left( \frac{2}{1 - \zeta_r} \right)^n \right)
\]

as a 2-adically convergent sum of integers.

The following result, which contains the conjectures of Adelberg [1] concerning the Nörlund numbers, may be derived from Theorem 3 by a method similar to that of §3:

**Theorem 4.** For all nonnegative integers \( n \), we have

\[
(-2)^n \frac{B_n^{(n)}}{n!} \equiv -1 + L_n 2^{[n/2]+1} + 2^{E_n} \pmod{2^{E_n+1} \mathbb{Z}_2}
\]

where \( L_n \) is given by

\[
L_n = \begin{cases} 
1, & \text{if } n \equiv 3, 4, 5 \pmod{8}, \\
-1, & \text{if } n \equiv 0, 1, 7 \pmod{8}, \\
0, & \text{if } n \equiv 2, 6 \pmod{8}.
\end{cases}
\]

\( E_3 = \infty \), and for \( n \neq 3, E_n \) is given by

\[
\begin{array}{cccccccc}
\text{Ord} & 8k - 2 & 8k - 1 & 8k & 8k + 1 & 8k + 2 & 8k + 3 & 8k + 4 & 8k + 5 \\
E_n & 6k & 6k + 1 & 6k + 2 & 6k + 2 & 6k + 3 & 6k + 5 & 7k + 6 & 6k + 5.
\end{array}
\]
The values of $L_n$ were called the conjectured stable values by Adelberg ([1], eq. (19)), who also observed the values of $E_n$ via numerical computation ([1], eq. (20)).

We give a brief outline of the proofs of these two theorems. If we divide both sides of (2.3) by $1 - 2t$, we get

$$\frac{-2t}{(1 - 2t) \log(1 - 2t)} - \frac{1}{1 - 2t} = \sum_{r=1}^{\infty} \frac{2^{r+1}t}{(1 - 2t)((1 - 2t)^{2^r} + 1)}$$

as an identity in $\mathbb{Z}_2[[t]]$. In the partial fraction decomposition of the rational functions in (4.1), the terms with denominator $1 - 2t$ all cancel (the numerator is $2 + \sum_{r=1}^{\infty} 2^r$, which is zero in $\mathbb{Z}_2$), since the singularity at $t = 1/2$ in the left-most term in (4.5) is removable. Thus we are left with

$$\frac{-2t}{(1 - 2t) \log(1 - 2t)} + \frac{1}{1 - t} = \sum_{r=1}^{\infty} \frac{2^r((1 - 2t)^{2^r-1} + 1)}{(1 - 2t)^{2^r} + 1},$$

which we rewrite as

$$\sum_{n=0}^{\infty} C_n t^n = \sum_{r=1}^{\infty} \sum_{n=0}^{\infty} C_{r,n} t^n$$

where $C_n = (-2)^n B_n^{(n)} / n! + 1$ as in [1] and, by partial fraction decomposition,

$$C_{r,n} = -\sum_{\zeta} \left(\frac{2}{1 - \zeta}\right)^n = -\text{Tr}_r \left(\left(\frac{2}{1 - \zeta_r}\right)^n\right),$$

where the sum is over all primitive $2^{r+1}$-th roots of unity $\zeta = \zeta_r$. Thus Theorem 1 implies Theorem 3. In fact the two are equivalent: Assuming Theorem 3, by (1.10) we have

$$(-2)^n b_n = (-2)^n \frac{B_n^{(n)}}{n!} - 2(-2)^{n-1} \frac{B_{n-1}^{(n-1)}}{(n - 1)!}$$

$$= -\sum_{r=0}^{\infty} \text{Tr}_r \left(\left(\frac{2}{1 - \zeta_r}\right)^n - 2 \left(\frac{2}{1 - \zeta_r}\right)^{n-1}\right)$$

$$= -\sum_{r=0}^{\infty} \text{Tr}_r \left(\zeta_r \left(\frac{2}{1 - \zeta_r}\right)^n\right),$$

proving Theorem 1. Alternately, Theorem 3 may also be derived from Theorem 1 via (1.10).

To prove Theorem 4, we observe that the sequence $\{C_{r,n}\}_{n=0}^{\infty}$ satisfies the same linear recurrence as $\{c_{r,n}\}_{n=0}^{\infty}$, and the lemma of §2 remains valid with $c_{r,n}$ replaced by $C_{r,n}$. Theorem 4 then follows from the following three assertions:
(1). \[ C_{1,n} = L_n 2^{[n/2]} + 1 \] for all \( n \);

(2). \( \text{ord} \left( C_{2,n} + C_{3,n} \right) = E_n \) for all \( n \neq 3 \);

(3). \( \text{ord} \ C_{r,n} > E_n \) for all \( r \geq 4 \) and all \( n \neq 3 \).

The value \( E_3 = \infty \) may be determined by direct calculation (cf. [1], p. 55). The analogue of (3.2) for the \( C_n \) is

\[
C_{1,n} = -((1 - i)^n + (1 + i)^n) = \begin{cases} 
-2(-4)^k, & \text{if } n = 4k, \\
-2(-4)^k, & \text{if } n = 4k + 1, \\
0, & \text{if } n = 4k + 2, \\
-(-4)^{k+1}, & \text{if } n = 4k + 3,
\end{cases} \tag{4.6}
\]

which proves Assertion (1), and the analogue of (3.4) is

\[
C_{r,n} = -\text{Tr}_r(\alpha^n) = -\sum_{\zeta} \alpha^n = -\sum_{\alpha} |\alpha|^n (\omega^n + \bar{\omega}^n) \tag{4.7}
\]

where \( \alpha = 2/(1 - \zeta), \omega = \alpha/|\alpha| \), the first sum is over all primitive \( 2^{r+1} \)-th roots of unity \( \zeta = \zeta_r \), and the second sum is over all such values of \( \alpha \) with positive imaginary part. This shows that \( C_{r,n} = 0 \) when \( n \equiv 2^r \pmod{2^{r+1}} \), so in particular \( C_{2,8k+4} = 0 \) for all \( k \). The recurrence

\[
a_k = -2176a_{k-1} - 4096a_{k-2} \]

for the lacunary subsequences \( \{a_k\} = \{C_{2,n+8k}\} \) easily shows that \( \text{ord} C_{2,n} = E_n \) for all \( n \neq 4 \pmod{8} \). The demonstration that \( \text{ord} C_{3,8k+4} = E_{8k+4} \) is a bit easier than the corresponding result for \( \epsilon_{3,8k-2} \), because the analytic function \( F(k) \) one obtains in the calculation analogous to (3.15) does not vanish at zero; this is why the formulas for \( E_{8k+j} \) do not depend on \( \text{ord} k \), as do those for the \( j = 0, -1, -2 \) cases of \( \epsilon_{8k+j} \).

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